Cambridge
Intermediate Mathematics

GEOMETRY

PART I

With Answers

Cambridge University Press
Fetter Lane, London

New York
Bombay, Calcutta, Madras
Toronto
Macmillan

Tokyo
Maruzen-Kabushiki-Kaisha

Cambridge Intermediate Mathematics

by

HERBERT JAMES LARCOMBE
PH.D., M.A., B.SC.

GEOMETRY

PART I

With Answers

CAMBRIDGE
AT THE UNIVERSITY PRESS
1928

PRINTED IN GREAT BRITAIN

PREFACE

The Cambridge Intermediate Mathematics series, consisting of text-books in Arithmetic, Algebra and Geometry, each in two parts, has been designed to meet the needs of pupils in the newly organised Modern Schools and Senior Classes recommended by the Hadow Report.

"The first work of teachers and administrators is," to use the words of *The New Prospect in Education*, just published by the Board of Education, "to think out their goal, feeling their way towards an appropriate curriculum." It is impossible to predict with any certainty the form which the curriculum will ultimately take; indeed, it is improbable that any such limitations as are imposed upon the Secondary School can ever be applied to the diversified types of senior schools which are about to spring up. But probably all are agreed that these schools must not become "an anaemic reflection of the present Secondary School."

The underlying notion on which the treatment in the books of this series has been based is that the aim of the Modern Schools, whether selective or non-selective, is to fit the pupils to take their places in the industrial and commercial rather than in the professional walks of life. For the latter the academic course of the Secondary School is a more or less fitting preparation; for the former it is decidedly out of place. It is assumed that the mathematical work of the newly organised schools will have a practical bias; their scholars need to be able to apply principles rather than to be able to derive them. Hence in these books theoretical explanations have been reduced to a minimum, and the use of the results has been emphasised.

GEOMETRY, PART I, deals in a simple manner with the more elementary principles of the subject, emphasising throughout the portions which lead directly to problems involving practical constructions and numerical calculations. It has not been thought necessary to deviate widely from the generally accepted sequence of proofs, but it is not intended that the proofs themselves shall be memorised. This book, with PART II, will introduce the pupil

to the elementary principles on which the more advanced parts of Geometry and Trigonometry are based; if he is transferred at any stage of the course to a Secondary School, or if he takes up his studies again in later life, he will not feel at a loss. PART I will provide a complete course of Geometry for the average non-selective senior school, and may even be found to be sufficient as a preparation for the simpler examinations.

A feature of the book is the introduction of exercises which can be solved either without any working on paper or with the aid of a rough diagram; for want of a better term these exercises have been described as *Mental*. The author believes that there is as much justification for mental geometry as for mental arithmetic, and for the same reason, namely to fix firmly and to revise rapidly the basic principles.

A further feature is the interpolation of a series of sectional revision exercises, each consisting of a *Mental* and a *Written* section; a general revision exercise concludes the book. These two features have also been adopted in the companion volumes of the series.

I am indebted to Mr E. F. Partridge, B.Sc., for his valuable assistance in obtaining solutions to the questions, to the Cambridge Local Examinations Syndicate for permission to use four pages from their *Cambridge Four-Figure Mathematical Tables*, and to the following authorities for their courtesy in allowing me to make use of examination papers: the Oxford Local Examinations Delegacy, the Cambridge Local Examinations Syndicate, the Royal Society of Arts, the East Midland Educational Union (E.M.E.U.), the Union of Educational Institutions, and the Union of Lancashire and Cheshire Institutes. Special attention is directed to the first papers set in the Royal Society of Arts Junior Schools Certificate Examination and in the E.M.E.U. Central Schools Examination, both instituted in 1927; copies of these papers will be found at the end of the book.

H. J. L.

September 1928

CONTENTS

GEOMETRY

PART I

1. Suppose you wanted to fix a position on one of the picture-rails of a room. You could do this by starting at either end of the rail and making *one measurement* from the end to the position in question.

If, however, you wished to fix a point on the floor of the room, it would be necessary to make *two measurements*; you could do it by starting at the point and measuring "straight" to a wall, and then from that point on the wall to a corner of the floor.

Finally, suppose you wanted to fix a point anywhere within the room, that is neither on wall, floor nor ceiling. In this case *three measurements* would be needed: you could measure from the point to be fixed straight to the floor, then from that point on the floor straight to a wall, and lastly from that point to the corner of the floor.

All objects existing by themselves in nature have three dimensions. You can imagine a *surface* of a "solid," but it is difficult to imagine a surface existing by itself, for it would have no thickness at all. Even the thinnest piece of paper has some thickness; you can calculate it by placing some hundreds of sheets together and dividing the total thickness by the number of sheets.

Still less can a *line* exist as an object by itself: the "line" drawn on paper is really a solid having length, breadth and thickness. Our "line" drawn on paper is the nearest representation we can give to it.

Even less still can a *point* exist by itself: the so-called point fixed with a pencil point is really a collection of particles of blacklead each having three dimensions.

This should give you some idea of:

(1) *No dimensions*—a point,

(2) *one dimension*—a line,

(3) *two dimensions*—a surface,

(4) *three dimensions*—an object or a space having volume.

2. It will be assumed that you are already acquainted with such "solids" as can be constructed out of cardboard or plasticine, e.g. the cube, rectangular solid, pyramid, cylinder, cone, and sphere. Each of these has surfaces, sometimes curved, sometimes flat. Some of the flat surfaces are rectangles, others triangles, pentagons, hexagons, circles and so on. All the surfaces except the sphere have edges, sometimes straight, sometimes curved. In most of the solids the edges meet in "corners" or points.

You will understand then that:

When two lines meet, they meet at a *point*.
Two surfaces meet in a *line*.

Lines.

Lines may be straight or curved. They may be tested by placing a "straight-edge" against them: if the line is straight, there will be no space between it and the straight-edge placed against it.

Surfaces.

Surfaces may be plane or curved, e.g. all the surfaces of a rectangular solid or pyramid are plane, whereas in a cylinder one surface is curved and two are plane.

Surfaces are bounded by lines, sometimes curved and sometimes straight, e.g. a circle has a boundary of one curved line, a triangle a boundary of three straight lines, a semi-circle a boundary of one curved and one straight line.

Note that what is generally called a circle is merely the circumference of a circle, and when we say, "Describe a semi-circle on a given straight line," we really mean, "Describe a semi-circumference."

EXERCISE 1.

1. Draw a line OX of any length. Along it measure off OA 2·3″, AB 1·1″, BC 1·3″. Measure the lengths of OC and AC.

Do the same as above with

2. OA 3·1″, AB 1·3″, BC ·9″. 3. OA 1·8″, AB 1·5″, BC 1·7″.

4. OA $1\frac{7}{8}$″, AB $1\frac{3}{4}$″, BC $1\frac{5}{8}$″. 5. OA $3\frac{1}{8}$″, AB $1\frac{1}{4}$″, BC $1\frac{9}{16}$″.

6. OA $2\frac{1}{2}$″, AB $1\frac{5}{12}$″, BC $1\frac{1}{12}$″.

By drawing a line and measuring off suitable distances upon prove that:

7. $5{\cdot}1 - 1{\cdot}4 = 3{\cdot}7$.

8. $2\frac{5}{6} - 1\frac{1}{4} = 1\frac{7}{12}$.

9. $1\frac{1}{2} + 1\frac{1}{3} + 1\frac{1}{4} = 4\frac{1}{12}$.

10. $2\frac{1}{8} + 1\frac{5}{8} + 1\frac{3}{8} = 5\frac{1}{8}$.

Draw the straight lines of which the dimensions are given in :hes, and find their lengths in centimetres to the nearest first ,ce of decimals:

11. $2{\cdot}4''$.

12. $1{\cdot}9''$.

13. $2{\cdot}1''$.

14 $3\frac{5}{16}''$.

15. $2\frac{7}{8}''$.

16. $1\frac{5}{12}''$.

Draw the straight lines of which the dimensions are given in ıtimetres etc., and find their lengths in inches to the nearest ;t place of decimals:

17. 9·5 cm.

18. 8 cm. 9 mm.

19. 12 cm. 1 mm.

20. 156 mm.

21. 108 mm.

22. 139 mm.

Using the relation 1 cm. = ·3937″ calculate:

23. The number of feet and inches in 256 cm., to the nearest inch.

24. The number of cm. and mm. in 5·97″, to the nearest mm.

25. The number of cm. and mm. in $1\frac{3}{16}''$, to the nearest mm.

Make rough diagrams showing:

26. A rectangular solid.

27. A triangular pyramid.

28. A crystal composed of two triangular pyramids base to base.

29. A crystal composed similarly of two square pyramids.

30. A pyramid on a base of 5 equal sides.

31. A cone.

32. A cylinder.

33. A hemisphere.

34. A cylinder surmounted by a cone.

Describe the surfaces, and give the number of each:

35. A cube.

36. A rectangular solid on a square base.

37. A regular square pyramid.

38. A cone.

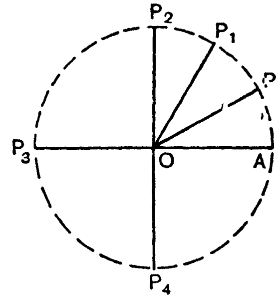
39. A cylinder.

40. A sphere.

Angular Measurement.

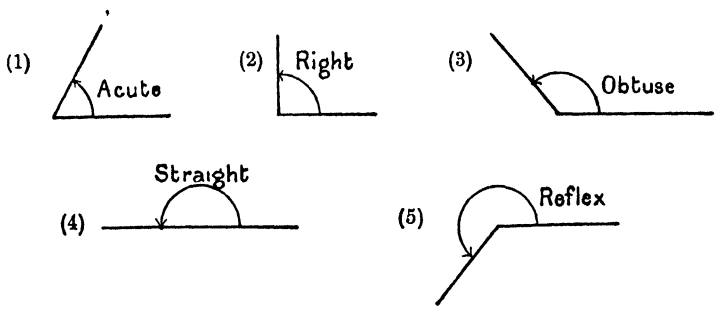
3. The idea of a *right angle* is familiar to everyone, though not everyone can satisfactorily define it. Some will tell you that it is an angle of 90°; others that it is the angle formed when one line is perpendicular to another. Neither is a definition: each is merely another description.

Suppose a line OA revolves in a direction opposite to that taken by the hands of a clock, i.e. in a counter-clockwise or anti-clockwise direction. It takes in succession the positions OP, OP_1, OP_2, OP_3, OP_4, etc., and returns to the original position OA.



It has performed one complete revolution, and has clearly traced out four right angles. This gives us material for a definition: *a right angle is the angle traced out by a line revolving round its extremity after a quarter of a complete revolution.*

4. Anything less than a right angle is called an *acute* angle, whilst an angle less than two right angles but more than one is called an *obtuse* angle. An angle of two right angles is called a *straight* angle. Finally an angle greater than two right angles but less than four is called a *reflex* angle.



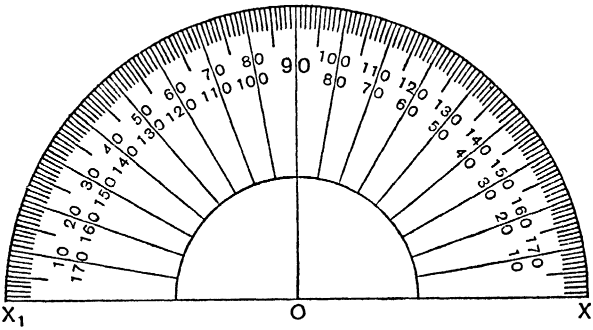
The complete revolution, then, gives four right angles, and this angle is divided up into 360 equal parts called *degrees.* Hence a right angle is one of 90 degrees—always written

90°. A straight angle is 180°. An obtuse angle is more than 90° but less than 180°. A reflex angle lies between 180° and 360°.

5. Two angles together equal to a right angle are said to be *complementary* to one another, and each is said to be the *complement* of the other: e.g. 30° and 60° are complementary; so are $x°$ and $(90-x)°$.

The Protractor.

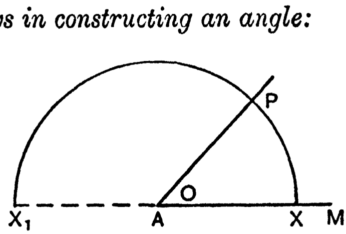
6. The *Protractor* is an instrument used in constructing and measuring angles. The following illustration shows one form of the instrument:



The Protractor is used as follows in constructing an angle:

Suppose $\angle MAP$ is to be 47°.

Place the protractor so that OX lies along AM, and O lies on the point A at which the angle is to be made.

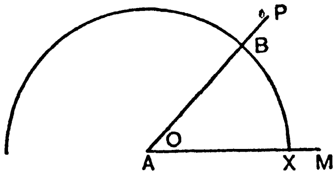


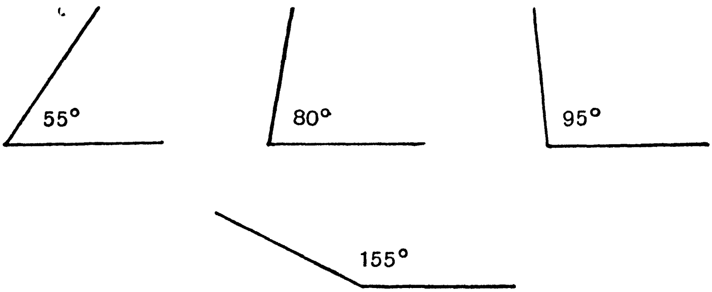
Count along the semi-circle from X until the 40° mark is reached, then count until 47° is found. Mark the position of the point P. Join AP. Then $\angle MAP$ is 47°.

In making an angle just less or just more than 90°, be careful to use the proper sequence of angles, or you may discover that you have made 83° instead of 97°, or vice versa.

The use of the Protractor in measuring an angle is as follows:

Placing the protractor so that O lies on A and OX lies along AM; note where AP cuts the semi-circle and read off the size of the angle. An eye estimate of the angle will always help you to decide whether you are using the appropriate sequence of angles, e.g.:



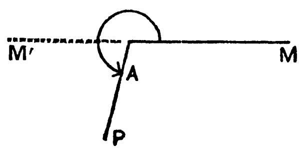


For *reflex* angles it is necessary to make an angle 180° less, with $M'A$ for one arm, as shown in the diagram.

E.g. suppose 257° is to be made.

$$257° - 180° = 77°.$$

$\therefore$ $M'AP$ must be made 77°, and the reflex MAP will then be 257°.



EXERCISE 2.

1. Draw an angle AOB of 37°. Produce BO to any point C. Measure $\angle AOC$ with protractor. Find the sum of $\angle$s AOB, AOC.

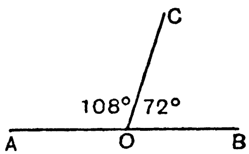
2. Make an angle AOB of 115°. Produce BO to any point C. Measure $\angle AOC$. Find the sum of $\angle$s AOB, AOC.

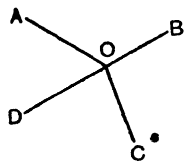
3. Draw any line BOC and another line AO meeting it at O. Measure $\angle$s AOB, AOC and find their sum.

4. Draw any line BOC with AO, DO, two other lines meeting it at O. Measure the $\angle$s BOD, DOA, AOC and find their sum.

5. Make $\angle AOC$ 108°, and $\angle BOC$ 72°.

What do you notice concerning the lines AO, OB?





6. Make $\angle AOB$ 121°, $\angle BOC$ 98°, $\angle COD$ 83°. Measure $\angle AOD$. Add to the total of the other three.

7. Draw any four lines as in the figure of Question 5. Measure the four angles and find their sum.

8. Through what angle does the minute hand of a clock pass in half an hour? Give the answer in two forms, right angles and degrees.

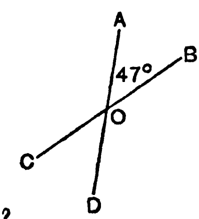
9. Through what angle does the hour hand of a clock pass in half an hour? Give the answer in two forms.

10. A line AB runs due E. and W., A being E. of B. AC is in a N.W. direction from A. What is $\angle CAB$ in degrees and also as a fraction of a right angle?

11. OA runs from W. to E. and B is N.N.W. of O. Find $\angle AOB$ in degrees.

12. What angle is formed between the directions S.S.W. and W.S.W. from a point?

13. Draw an angle AOB 47°. Produce AO and BO as shown in the figure. Measure $\angle$s AOC, COD, DOB. What do you notice?



14. Do the same with $\angle AOB$ 128°.

15. Do the same with $\angle AOB$ 90°.

16. Draw any two lines cutting one another and measure all four angles. What do you notice?

7. Summary of Observations.

1. If CO meets a straight line AOB at O,

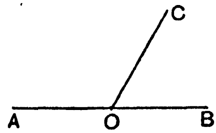
$$\angle COA + \angle COB = 2 \text{ right angles},$$

and $\angle$s AOC, BOC are said to be *supplementary*, each being the *supplement* of the other.

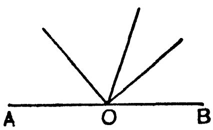
2. If CO meets two lines AO, OB so that

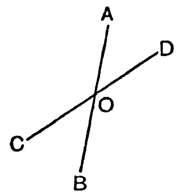
$$\angle COA + \angle COB = 2 \text{ right angles},$$

AOB is one straight line.



3. If any number of lines are drawn meeting a given line AB at one point the sum of the separate angles so formed is equal to two right angles.





4. When two straight lines AB, CD cut at O, the *vertically opposite* $\angle$s AOD, COB, and also $\angle$s AOC, BOD are equal.

It will be sufficient to indicate the proof for one angle AOD of 37°.

$$\begin{aligned}
&(1)\quad \angle COD = 180^\circ, \\
& \angle AOD = 37^\circ, \quad \therefore\ \angle AOC = 143^\circ. \\
&(2)\quad \angle AOB = 180^\circ, \\
& \angle AOC = 143^\circ, \quad \therefore\ \angle BOC = 37^\circ. \\
&(3)\quad \angle COD = 180^\circ, \\
& \angle BOC = 37^\circ, \quad \therefore\ \angle BOD = 143^\circ.
\end{aligned}$$

Learn:

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.

If a straight line meets two straight lines at the same point, so as to make the sum of the two adjacent angles so formed equal to two right angles, the two straight lines are in the same straight line.

If two straight lines intersect, the vertically opposite angles are equal.

Simple Constructions.

8. To bisect a given line.

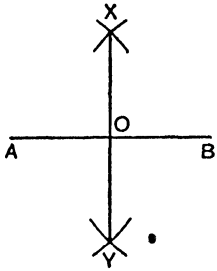
1. Let AB be the line to be bisected.

With centre A and any convenient radius describe arcs above and below the line.

With centre B and the same radius cut those arcs at X and Y.

Join XY.

This cuts AB at O, and AB is bisected at O. No proof is required at this stage.



2. If the length of the line is given, a distance AO equal to half of this can be cut off from A.

3. If tracing paper is used, the paper can be folded so that A coincides with B, and the fold will pass through the point of bisection.

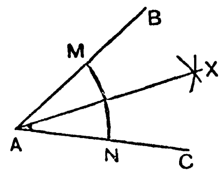
9. To bisect a given angle.

1. Let BAC be the angle to be bisected.

With centre A and any radius describe an arc cutting AB, AC at M and N.

With centres M and N and any radius describe arcs cutting at X.

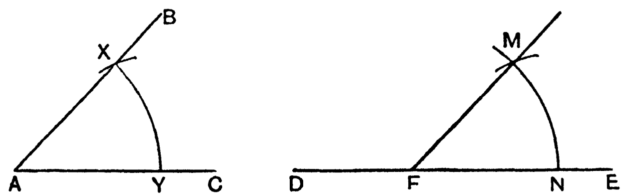
Then AX bisects $\angle BAC$.



2. If the number of degrees in $\angle BAC$ is known, an angle BAX equal to half of this can be constructed with the protractor.

3. If tracing paper is used, the paper can be folded through A so that AB coincides with AC, and the fold will bisect the angle.

10. To copy a given angle.



Let BAC be the angle to be copied at F in the line DE.

With centre A and any radius describe an arc cutting AB and AC at X and Y.

With centre F and the same radius describe the arc MN.

Measure the distance YX.

Cut off NM equal to it and join FM.

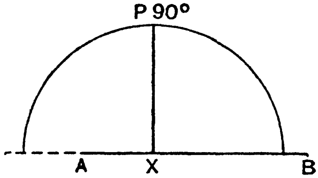
Then $\angle MFN = \angle BAC$.

11. Construction of Perpendiculars.

1. From a given point in a line.

(1) *With protractor.*

The protractor is placed as shown in the figure and XP is drawn to the point indicating 90°.

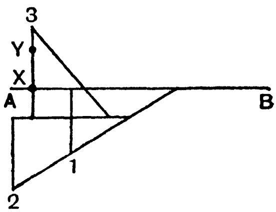


(2) *With set-squares*; this is the commonest method.

One of the set-squares is placed against the line as shown by the sketch of the set-square with vertex 1.

We then slide this set-square along a straight-edge, away from AB, until the vertex takes any other position 2.

The second set-square is then placed as indicated by the triangle with vertex 3, the edge passing through X.



Note that this method is also used when the given point Y is outside the line.

(3) *With ruler and compasses.*

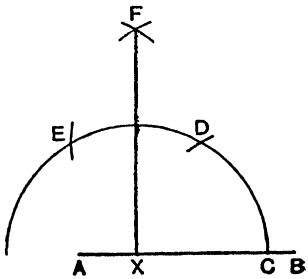
With centre X and any radius make an arc cutting AB at C.

With centre C and the same radius cut this arc at D.

With centre D and the same radius make arcs at E and F.

With centre E and the same radius cut the former arc at F.

Join FX.



The reason will be seen later. We are actually making an angle DXC 60°, and DXE 60°, and bisecting $\angle DXE$.

$$\therefore \angle FXC = 60° + \tfrac{1}{2} \text{ of } 60°$$
$$= 60° + 30°$$
$$= 90°.$$

2. From a point outside the line.

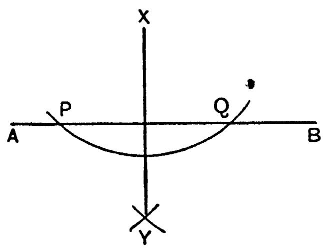
12. (1) *With set-squares.* See previous section.

(2) *With ruler and compasses.*

With centre X and any suitable radius cut the line AB at P and Q.

With centres P and Q describe arcs cutting at Y.

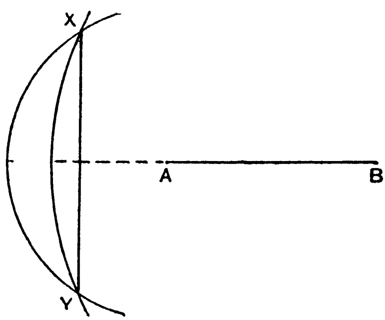
Join XY. This is the required perpendicular.



(3) *Second method with ruler and compasses.*

With centres A and B describe arcs through X cutting again at Y.

Join XY.



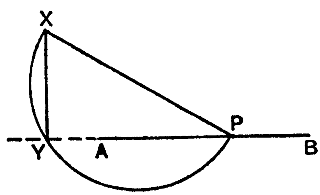
(4) *Third method with ruler and compasses.*

Join X to any point P in the line AB.

On XP describe a semi-circle cutting AB at Y.

Join XY.

Methods (3) and (4) can be used in certain cases where method (2) would be impossible.

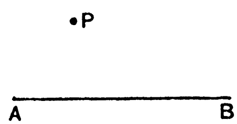


13. Parallel Straight Lines.

Though you may never have attempted to give a definition of parallel lines, you will be able to give many examples of such *pairs of lines* from objects you see about you; e.g., opposite edges of a page, a picture frame, a ceiling, a wall, or a ruler. The essential feature of parallel lines is that they keep the same distance apart, and hence never meet.

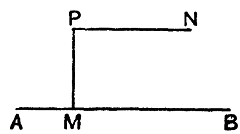
The following are three methods of drawing a line parallel to another through a given point:

Method 1.



Let AB be the given line and P the given point outside. The parallel to AB is to be drawn through P.

It is well known that the opposite sides of a rectangle are parallel, and this is assumed in the method which follows:

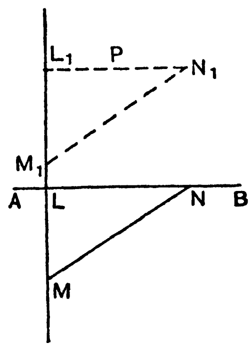


Draw PM at right angles to AB by one of the methods shown in the previous paragraph.

Then draw PN at right angles to PM.

We now have three sides of a rectangle, and PN is parallel to AB.

Method 2.



Place a straight-edge in such a position that one edge of a set-square lies against it at LM, and another edge lies along the line AB at LN.

Slide LM along the straight-edge until the line LN passes through P as at L_1N_1.

Then L_1N_1 will be parallel to AB.

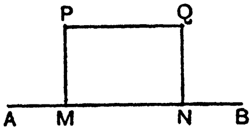
The proof will be easy to discover later.

Method 3.

Draw PM perpendicular to AB.

Draw another perpendicular QN of the same length.

Then PQ will be parallel to AB.



EXERCISE 3.

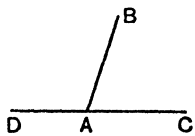
1. Draw a line 3·25″ long and bisect it.

2. Divide a line 9·3 cm. long into four equal parts by the use of ruler and compasses.

3. Make an angle of 35° and bisect it.

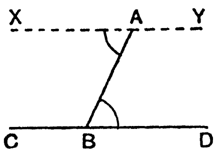
4. Divide an angle of 127° into four equal parts by the use of ruler and compasses.

5. Construct this figure with $\angle BAC$ 73°.
Bisect both angles at A and measure the angle between the bisectors AX, AY.

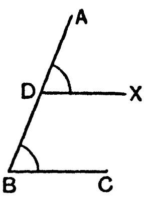


6. In Question 5, how could you prove that $\angle XAY$ is a right angle?

7. Draw any parallelogram $ABCD$ and bisect the angles A and B. Let the bisectors be AX and BX. Measure $\angle AXB$.



8. Make any angle ABD, and produce DB to C.
At A make $\angle XAB$ equal to $\angle ABD$, using ruler and compasses.
Measure $\angle$s YAB, ABC.



9. Make any $\angle ABC$. In AB take a point D.
At D make $\angle ADX$ equal to $\angle ABC$.
What do you notice about DX and BC?

10. Draw any triangle ABC. Make another triangle with the same angles on a base XY, by the use of ruler and compasses.

11. Draw any line AB and fix a point P about 1″ away from it. Through P draw a parallel to AB in three different ways:

(*a*) By drawing PM perpendicular to AB with a set-square and making a right angle at P.

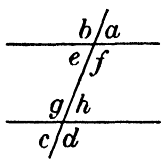
(*b*) By sliding a set-square.

(*c*) By drawing a second perpendicular to AB equal to PM.

12. Draw any line AB and draw another parallel to it and 1″ away. How many such lines could you draw? [Note. First draw a perpendicular to AB 1″ long.]

13. Draw any two lines AB, CD cutting one another at P. Draw 2 lines parallel to AB and 1″ away, and 2 lines parallel to CD and 1″ away.

14. Draw two lines AB, AC making an angle of 43°. Using the method of Question 3, find *one* point $1\frac{1}{2}$″ away from both lines. How many such points could be found, if BA and CA were produced?



15. Draw any two parallel lines and a line cutting them, as shown in the diagram.

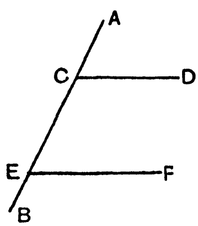
Measure angles *a* and *h*.

What do you find?

16. Measure angles *d* and *f*. 17. Measure angles *c* and *e*.

18. Measure angles *b* and *g*.

19. Draw any line AB.



At any point C make the $\angle ACD$ 38°.

At any other point E make the $\angle CEF$ 38°.

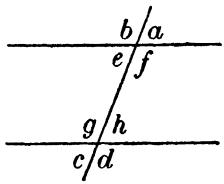
Place one set-square with an edge along EF, and a second set-square along another edge of the first.

Slide the first up until it reaches CD.

What do you find?

20. Repeat this with an $\angle ACD$ of 102°, and CEF 102°.

14. Observations.



When a straight line cuts any two straight lines as shown in the figure, the following pairs of angles are said to be *corresponding* angles:

a, h $\qquad b, g$

c, e $\qquad d, f.$

Note that (1) a is *exterior*, h *interior*.

(2) They are on the *same* side of the cutting line.

(3) They are *not adjacent* to one another.

In some of the examples just worked you should have discovered that, when the two lines are parallel,

$$a=h; \quad b=g; \quad c=e; \quad d=f.$$

That is, the corresponding angles are equal.

In others you should have discovered that if the lines were drawn such that either

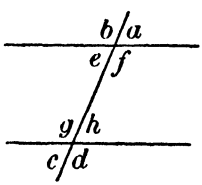
$$a=h, \text{ or } b=g, \text{ or } c=e, \text{ or } d=f,$$

the two lines cut by the *cutting line*, or *transversal*, are parallel.

Learn:

1. ***If a straight line cuts two parallel straight lines, the corresponding angles are equal.***
2. ***If a straight line cuts two straight lines, so as to make a pair of corresponding angles equal, the two straight lines are parallel.***
3. ***Corresponding angles are such that:***
   One is exterior and the other interior.
   They are on the same side of the cutting line.
   They are not adjacent.

EXERCISE 4.



1. Draw two parallel lines and any line cutting them.

Measure and compare angles e and h.

What do you notice?

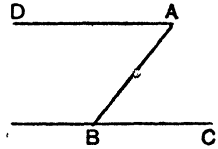
2. Measure f and g. Do you find a corresponding result?

3. Make an angle ABC of 49°.

At A make an angle DAB 49°.

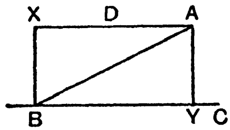
4. Repeat this with an angle ABC of 127°.

Test DA and BC with set-squares and see if they are parallel.



5. Repeat this with an angle ABC of 90°.

6. Repeat with an angle ABC of 28°.

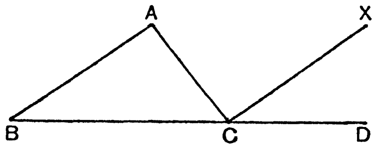


Draw perpendiculars BX and AY and measure them.

7. Draw ABC of any dimensions and produce BC to D.

Draw CX parallel to BA.

Measure ∠s BAC, ACX and compare them.



Measure ∠s ABC and XCD and compare them.

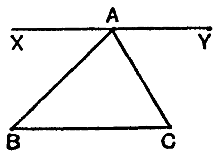
8. Draw any triangle ABC.

Draw XAY parallel to BC.

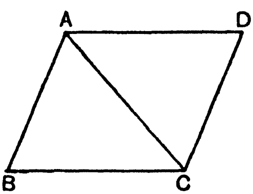
Measure and compare ∠s XAB, ABC.

Also compare ∠s YAC, ACB.

What do you notice?



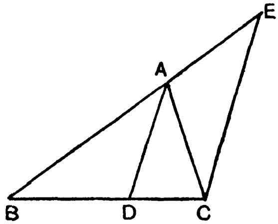
9. Draw any parallelogram $ABCD$.



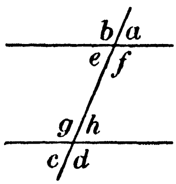
Measure and compare ∠s BAC, ACD.

Also measure and compare ∠s DAC, ACB.

10. Draw any triangle ABC.
Bisect $\angle BAC$ by AD.
Draw $CE \parallel$ to AD.
Measure $\angle$s ACE, AEC and compare with $\angle$s BAD, DAC.



15. Observations.



When a straight line cuts any two straight lines as shown in the figure,

Angles e and h are said to be *alternate*;

Angles f and g are also *alternate*.

I.e. Alternate angles are two *non-adjacent interior* angles on *opposite sides* of the cutting line.

When two straight lines are parallel we might have expected to find that

$$\angle e = \angle h,$$
$$\angle f = \angle g.$$

For we have proved $\quad \angle a = \angle h.$

We know $\quad \angle a = \angle e.$

$$\therefore \angle e = \angle h,$$

and similarly $\quad \angle b = \angle g,$

$$\angle b = \angle f.$$
$$\therefore \angle f = \angle g.$$

The converse is also true.

If $\angle e$ is known to be equal to $\angle h$, since we know $\angle e = \angle a$,

$$\therefore \angle a = \angle h.$$

But these are corresponding angles.

Therefore the lines cut by the *transversal* (or cutting line) are parallel.

Learn:

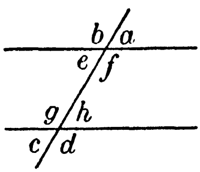
1. *If a straight line cuts two parallel straight lines, the alternate angles are equal.*
2. *If a straight line cuts two straight lines so as to make the alternate angles equal, the two straight lines are parallel.*
3. *Alternate angles are such that:*

   Both are interior.

   They are on opposite sides of the cutting line.

   They are not adjacent.

EXERCISE 5.

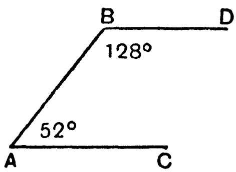


1. Draw any two parallel lines, and a line cutting them. Measure angles f and h with the protractor. Find their sum.

2. Measure angles e, g and find their sum.

3. Repeat with the cutting line at right angles to the parallels.

4. Make an angle BAC of 52°.



At B make an angle ABD of 128°.

Note that $128° + 52° = 180°$.

See if AC, BD are parallel by using set squares.

5. Repeat with angles of 140°, 40°.

6. Repeat with angles of 90°, 90°.

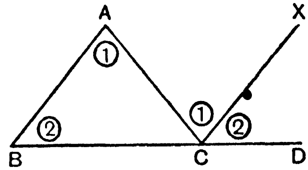
Learn:

If a straight line cuts two parallel straight lines, the two interior angles on the same side of the cutting line are together equal to two right angles.

If a straight line cuts two straight lines so as to make two interior angles on the same side of the cutting line equal to two right angles, the two straight lines are parallel.

ANGLES OF TRIANGLES AND RECTILINEAL FIGURES

16. Relation between exterior angle and interior angles of a triangle.



$\angle ACD$ is *exterior*.

$\angle ACB$, $\angle ABC$, $\angle BAC$ are *interior*.

We can prove that

$$\angle ACD = \angle ABC + \angle BAC.$$

Proof.

Draw $CX \parallel$ to AB.

The angles marked (1) are equal alternate angles; those marked (2) are equal corresponding angles.

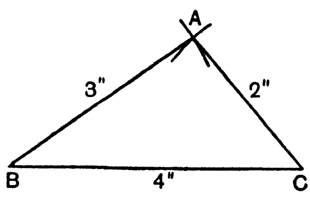
$$\therefore \angle ACD = \angle ABC + \angle BAC.$$

Learn:

The exterior angle of a triangle is equal to the sum of the two interior opposite angles.

Hence it is clear that an exterior angle of a triangle is greater than either of the two interior opposite angles.

17. Construction of Triangles given the lengths of three sides.



Suppose the sides are 3″, 2″, 4″.

Let ABC be a rough figure. On it mark BC 4″, CA 2″, AB 3″.

Now you are ready to start the construction. Make BC 4″. With the point of the compasses on B and with a radius 3″, obtained by measuring from a ruler, make an arc. With the point of the compasses on C and with a radius 2″, make another arc cutting the first at A. Join AB, AC. Then the triangle ABC has been made with AB 3″, BC 4″, CA 2″.

EXERCISE 6.

1. Make a triangle ABC in paper or cardboard.

Cut it into 3 portions as shown in Fig. 1.

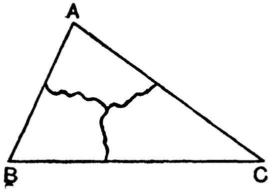


Fig. 1

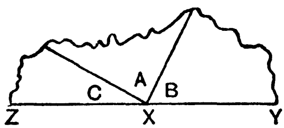


Fig. 2

Fit the angles together as shown in Fig. 2.

What do you notice about the new arms XY, XZ?

What does this show?

2. Make any triangle and measure its angles with the protractor. Find the sum.

3. Make a triangle with sides $3''$, $4''$, $4\frac{1}{2}''$. Measure the angles and find the sum.

4. Repeat with sides $4''$, $4''$, $4\frac{1}{2}''$. Find the sum. What do you notice besides the total value of the angles?

5. Repeat with sides $3''$, $3''$, $3''$. Find the sum. What else do you notice?

6. If two angles of a triangle are $50°$ and $70°$, what is the third angle?

7. If two angles of a triangle are $103°$ and $43°$, what is the third angle?

8. Try to make a triangle with sides $3''$, $2''$ and $5''$. What do you discover?

9. Try to make a triangle with sides $3''$, $2''$, $6''$.

10. If the angles of a triangle are represented by $x°$, $2x°$ and $3x°$, find the size of each angle.

11. If one angle of a triangle is $24°$ and the other two are equal, how many degrees are there in each?

12. Find the angles of a triangle if each of the angles at the base is double the vertical angle.

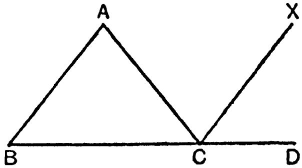
13. Find the angles of a triangle if each angle at the base is half the vertical angle.

14. ABC is a triangle with one side BC produced to D.

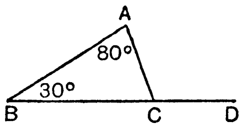
CX is drawn parallel to AB.

By finding a pair of equal corresponding angles and a pair of equal alternate angles, prove

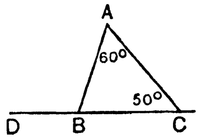
$$\angle ACD = \angle ABC + \angle BAC.$$



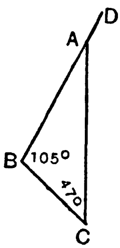
15. Calculate $\angle ACD$.



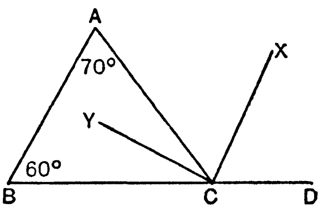
16. Calculate $\angle ABD$.



17. Calculate $\angle DAC$.



18. If CX, CY bisect the angles ACD, ACB calculate the angle XCY.

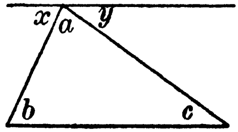


18. Angles of a Triangle.

In the preceding exercise you have used the fact that the sum of the angles of a triangle is equal to two right angles.

The following is a further proof:

Draw a line through the vertex parallel to the base.



Then $\angle x = \angle b$ *alternate,*

$\angle y = \angle c$ *alternate.*

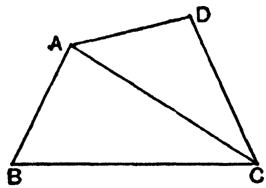
$$\therefore x + a + y = b + a + c.$$

$$\text{But } x + a + y = 2 \text{ rt} \angle \text{s}.$$

$$\therefore b + a + c = 2 \text{ rt} \angle \text{s}.$$

Learn:

The sum of the angles of a triangle is equal to two right angles.



19. (i) **Angles of a Quadrilateral,** i.e. a four-sided figure.

By dividing it into two triangles we see that all the angles of the quadrilateral are together equal to all the angles of the two triangles.

∴ the sum is 4 right angles.

Learn:

The sum of the angles of a quadrilateral is equal to four right angles.

(ii) **Angles of a Five-sided Figure,** i.e. a pentagon.

Here we can divide into 3 triangles.

∴ the sum is 6 right angles.

(iii) The following table will help us to arrive at a **general solution:**

Sides	Sum of angles	
3	2	or $2 \times 3 - 4$ rt $\angle$s
4	4	or $2 \times 4 - 4$ „
5	6	or $2 \times 5 - 4$ „
6	8	or $2 \times 6 - 4$ „
n	$2n - 4$ rt $\angle$s.	

20. Angle of a Regular Polygon.

Hence in a regular polygon each of the n angles is equal to $\frac{2n-4}{n}$ right angles, or to $\frac{90}{n}(2n-4)$ degrees.

Ex. 1. *Suppose there are* 12 *sides.*

$$\begin{aligned} 12 \text{ angles} &= 24 - 4 \text{ rt } \angle\text{s} \\ &= 20 \text{ rt } \angle\text{s} \\ &= 1800°. \\ \therefore \text{each } \angle &= 150°. \end{aligned}$$

Ex. 2. *Suppose there are* 240 *sides.*

Using formula:

$$\text{Each angle} = \tfrac{90}{240}\,(480 - 4) \text{ degrees}$$

$$= \frac{\overset{3}{\cancel{90}}}{\underset{\underset{2}{\cancel{8}}}{\cancel{240}}} \times \overset{119}{\cancel{476}} \text{ degrees}$$

$$= 178\tfrac{1}{2}^\circ.$$

Ex. 3. *Find how many sides there are in a regular polygon if each angle* = 135°.

$$135 = \frac{90}{n}(2n - 4).$$

$$135n = 180n - 360.$$

$$45n = 360.$$

$$n = 8.$$

EXERCISE 7.

1. Draw any quadrilateral and find the sum of its angles by measurement.

2. Three angles of a quadrilateral are 50°, 70° and 108°. Find the fourth angle.

3. Two angles of a quadrilateral are right angles and one of the others is 70°. Find the fourth angle.

4. The angles of a quadrilateral are x°, $2x^\circ$, $3x^\circ$, $4x^\circ$. Find the number of degrees in each.

5. The angles of a quadrilateral are $x + 20^\circ$, $x + 30^\circ$, $x + 50^\circ$ and 80°. Find x.

6. Can you make a quadrilateral with angles of 50°, 70°, 95°, 145°?

7. Draw any five-sided figure and produce the sides in order. Measure the exterior angles and find their sum.

8. Repeat this with a quadrilateral.

Find the number of right angles in the sum of the angles of the following figures:

9. A 10-sided figure.

10. A 24-sided figure.

In the following regular figures, i.e. having all sides and all angles equal, find the number of degrees in each angle:

11. 5 sides.

12. 6 sides.

13. 7 sides.

14. 10 sides.

15. 100 sides.

16. 120 sides.

Find the number of sides in the regular polygons having angles of

17. 90°.

18. 108°.

19. 150°.

20. $177\frac{3}{5}°$.

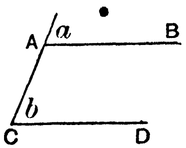
SECTIONAL REVISION A

EXERCISE 8 (*a*). MENTAL.

1. If a solid has 4 triangular faces, how many edges has it?

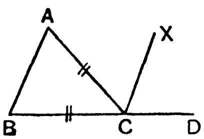
2. How many degrees are there in $\frac{2}{5}$ of a right angle?

3. Two angles of a triangle are 78° and 42°. What is the third?



4. Under what conditions are angles a and b equal?

5. How many triangles can be constructed on one side of a given line 2″ long, having two other sides of 1·9″ and 1·5″?



6. ABC is an isosceles triangle having CA equal to CB and BC is produced to D.
$\angle ACD$ is bisected by CX.
Write down all equal angles.

7. If from a line AB a distance 1·75 inches is cut off and the rest is found to be 2·35 inches long, what is the length of AB?

8. What is the size of the angle of a regular pentagon?

EXERCISE 8 (*b*).

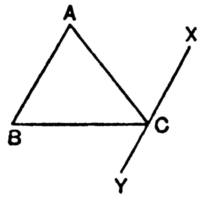
1. Draw a diagram representing a rectangular solid of which one vertical face is only just seen.

2. How many vertices has a pyramid on a 5-sided base? Draw a rough diagram to illustrate your answer.

3. Draw a line 5·7 cm. long and bisect it with the use of ruler and compasses.

4. Construct a triangle with sides 5·3 cm., 5·9 cm. and 7·3 cm. Measure the largest angle.

5. Draw any $\triangle\, ABC$ as shown. Through C draw XCY parallel to AB.



Prove that the three angles of your triangle are equal to two right angles.

6. Draw an angle BAC of 59°. Produce the arms BA and CA to D and E. Bisect angles BAC, DAE. What do you notice?

7. Draw any parallelogram and bisect two opposite angles. What do you notice about the bisectors?

8. Draw any triangle ABC and bisect all three sides, AB at X, BC at Y and CA at Z. Join CX, AY, BZ. What do you notice?

EXERCISE 9 (*a*). Mental.

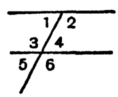
1. How many vertices has a cube?

2. What is the angle formed by the hands of a clock at 2 o'clock?

3. A quadrilateral has three of its angles right angles. Why must the fourth angle also be a right angle?

4. If two angles of a triangle are 85° 30′ and 45° 30′, what is the third?

5. Name the pairs of alternate angles in this figure.



6. If two adjacent angles, formed when one line meets another line, are equal, why must the two lines be at right angles to one another?

7. What is the complement of 28° 50′?

8. Why is it impossible to make a triangle with sides 4″, 2″, $1\frac{1}{2}$″?

EXERCISE 9 (*b*).

1. Construct a triangle with sides 6 cm., 8 cm., 10 cm. and measure the largest angle.

2. Draw AB 3″ long. Find a point X in it so that $AX = 1{\cdot}2''$. At X draw a perpendicular to AB, with the use of ruler and compasses.

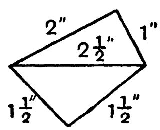
3. Make an angle of 117° with the protractor and bisect it.

4. Make a triangle with sides 2″, 3″, 4″, and draw perpendiculars from the vertices to the opposite sides. What do you notice?

5. One angle of a triangle is one-third of a right angle, and another is half a right angle. How many degrees are there in the third angle?

6. Draw a rough figure representing two regular tetrahedra on a common base. Name all its edges. How many vertices has this object?

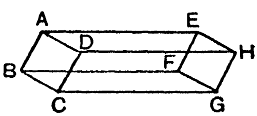
7. Construct the figure shown in the rough diagram.



8. A pavement is made up of regular tiles. Why will tiles with angles of 60°, 90°, 120° be suitable for this, and not a tile with an angle of 108°?

EXERCISE 10 (*a*). MENTAL.

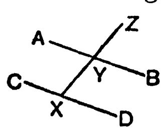
1. Name the parallelograms in this solid, assuming that each face is a parallelogram.



2. If in the parallelogram $ADHE$ two opposite angles are 120° each and the other two are equal, how large is each?

3. What is the angle between the hands of a clock at 5 o'clock?

4. Name two pairs of alternate angles in this figure.

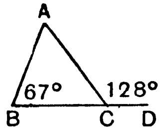


5. What is the supplement of $(120-x)^\circ$?

6. If from a line 9·05 cm. a distance 35 mm. is cut off, how many millimetres remain?

7. How many right angles are there in 1350 degrees?

8. What is the size of the angle BAC in this figure?



EXERCISE 10 (*b*).

1. Construct a triangle with two sides 8·3 cm. and one side 4·1 cm. From the ends of the base draw perpendiculars to the two sides.

2. Draw a line AB 3·1 in. long. Bisect it at right angles. On the right bisector take a point P 1·7″ from the line. Measure PA and PB.

3. If the angles of a quadrilateral are $(90-x)^\circ$, $(180-x)^\circ$, 60° and 130°, find the value of x.

4. How many sides are there in a regular polygon each angle of which is 144°?

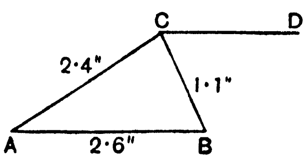
5. Construct a reflex angle of 315° with the help of the protractor.

6. Make a triangle with sides 3·1″, 2·8″, 2·5″.

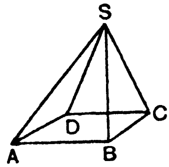
Construct a triangle with equal angles and with the longest side 1·7″.

7. In a line 3·9″ long fix a point 1·1″ from one end. At this point draw a perpendicular to the given line with the help of ruler and compasses.

8. Construct this figure in which CD is parallel to AB.



EXERCISE 11 (*a*). Mental.



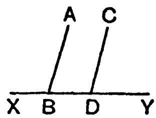
1. Write down the names of the edges in this solid.

2. If the above is a regular pyramid and $\angle BSC = 42°$, what is the size of $\angle SBC$?

3. If $SB = 3''$ and $SC = 3''$ and $\angle BSC = 42°$, construct the triangle SBC.

4. What angle does the minute hand of a clock traverse in 10 minutes?

5. If AB and CD are parallel, name a pair of supplementary angles which are not adjacent.



6. What is the complement of an angle of $\frac{1}{5}$ of a right angle in degrees?

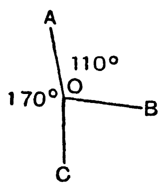
7. What is the size of each angle of a regular hexagon?

8. Why is it impossible to describe a triangle with angles of 105°, 75° and 25°?

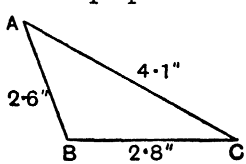
EXERCISE 11 (*b*).

1. Construct an equilateral triangle with sides 2·9 in. long. Bisect any one of the angles and measure the segments of the side cut by the bisector.

2. Construct this figure with arms of any length and measure $\angle BOC$.



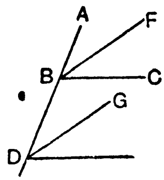
3. Construct this figure, and then with the help of ruler and compasses draw a line from A perpendicular to the direction of BC.

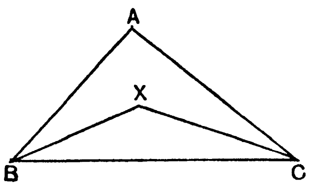


4. Make an angle of 37° with the aid of the protractor and then construct an angle of 217°, naming the angle so constructed, and indicating it with an arc.

5. Draw any parallelogram. Bisect two adjacent angles and measure the angle between the bisectors.

6. If $\angle ABC = \angle BDE$ and these angles are bisected by BF, DG, why is BF parallel to DG?





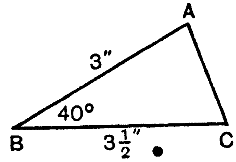
7. If $\angle ABC = 48°$, and $\angle ACB = 38°$, and these are bisected by BX and CX, calculate $\angle BXC$.

8. Draw a rough diagram showing a pyramid resting on a face of a cube.

CONGRUENCE OF TRIANGLES

21. 1st Case.

Suppose we have to construct accurately the triangle indicated in the rough figure.



Here we are given two sides and the included angle. We first make BC $3\frac{1}{2}''$ long.

Next make $\angle ABC$ 40° with protractor.

Make the arm BA 3″ long.

Join AC.

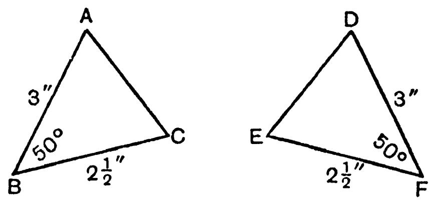
Then ABC is the triangle required.

Suppose all the triangles made correctly to these dimensions by a class were cut out and put together. It would be found that any two of them would fit exactly together. They are all said to be congruent.

Learn:

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.

Example. These are congruent because AB, BC and $\angle ABC$ are equal to DF, FE and $\angle DFE$.



Note that, if they are congruent, all parts must fit together.

$\therefore$ (1) $AC = DE$.

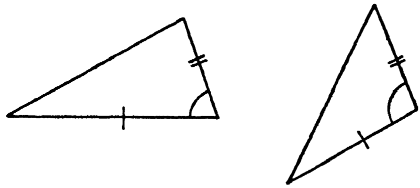
(2) $\angle ACB = \angle DEF$ ....opposite to 3″ sides.

(3) $\angle BAC = \angle EDF$ ....opposite to $2\frac{1}{2}''$ sides.

(4) Area of $\triangle ABC$ = area of $\triangle DEF$.

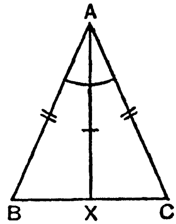
It is usual to mark off similarly on the rough figure the parts that are known to be equal, e.g.

The markings indicate 2 sides and the contained angles equal in each.



22. The following exercise gives an application of the principle.

Suppose ABC is a triangle with two sides equal, viz. $AB = AC$.



Let AX bisect the angle BAC, i.e. let $\angle BAX = \angle CAX$.

Now we have two triangles BAX, CAX in which

$$\begin{cases} AB = AC. \text{ Given.} \\ AX \text{ is the same in both } \triangle \text{ s.} \\ \angle BAX = \angle CAX. \end{cases}$$

These triangles have two sides and the contained angle equal in each. They are therefore congruent.

$$\therefore BX = XC \text{ .... } (1),$$
$$\angle ABX = \angle ACX \text{ .... } (2),$$
$$\angle AXB = \angle AXC \text{ .... } (3).$$

These three results, obtained when the vertical angle of an isosceles triangle is bisected, are worth examining.

(1) *The bisector of the vertical angle of an isosceles triangle also bisects the base.*

(2) *The angles opposite to the equal sides of an isosceles triangle are equal.*

(3) Since $\angle AXB + \angle AXC = 2$ rt $\angle$ s, and these are equal, it follows that each is a right angle. Therefore *the bisector of the vertical angle of an isosceles triangle is at right angles to the base.*

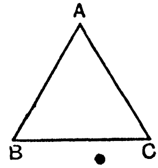
The second of these results must be known:

The angles at the base of an isosceles triangle are equal.

23. Equilateral Triangles.

Since the angles at the base of an Isosceles Triangle are equal, it follows that all the angles of an Equilateral Triangle are equal:

$$AB = AC,$$
$$\therefore \angle ABC = \angle ACB.$$
$$AC = BC,$$
$$\therefore \angle ABC = \angle BAC.$$
$$\therefore \angle A = \angle B = \angle C.$$

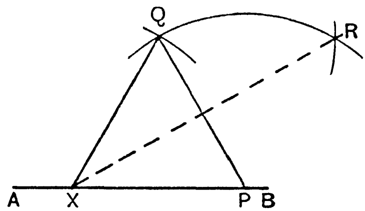


•Again, since $\angle A + \angle B + \angle C = 180°$,

$$\therefore \text{each angle} = 60°.$$

24. Geometrical Construction of angles of 60° and 30°, 120°, 150°.

The above gives us a method of constructing an angle of 60° at any point X in the line AB.



With centre X and any radius describe arcs at P, in AB, and Q.

With centre P and the same radius describe an arc cutting the former arc at Q.

Then QXP is an equilateral triangle and $\angle QXP = 60°$.

To construct 30°, take the same radius, and with P and Q as centres describe arcs cutting at R. Join XR.

Then $\angle RXP = 30°$.

$\angle AXQ = 120°$ and $\angle AXR = 150°$.

EXERCISE 12.

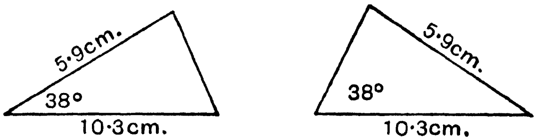
1. Make a triangle with one side $2\frac{3}{4}''$, a second side $3\frac{1}{4}''$ and a contained angle 49°. Measure the third side and the other two angles.

2. Repeat with sides 2·7″, 3·3″ and contained angle 102°.

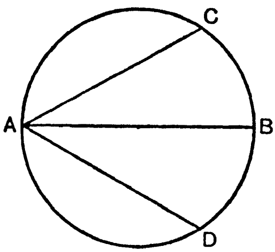
3. Repeat with sides 9·3 cm., 8·7 cm. and contained angle 45°.

4. Make a triangle ABC in which a (the side opposite to the angle A) $= 2{\cdot}5''$, b (opposite to the angle B) $= 2{\cdot}9''$ and the angle $C = 83°$. Make another $\triangle\ ABC$ in which $a = 2{\cdot}9''$, $b = 2{\cdot}5''$, $\angle C = 83°$. Are these congruent?

5. Are these triangles congruent? Give reasons. (The diagrams are purposely inaccurate.)



6. AC and AD are equal chords of a circle and AB bisects the $\angle CAD$.

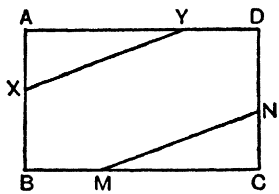


Show that the chords BC, BD are equal.

7. Show that the bisector of any angle of an equilateral triangle is at right angles to the opposite side.

8. Two right-angled triangles, i.e. triangles with one angle a right angle, have sides containing the right angles 8·7 cm. and 9·5 cm. Show that they are congruent.

9. $ABCD$ is a rectangle of which, of course, the opposite sides are equal.



AY and CM are cut off equal to one another, and $AX = CN$. Prove that $XY = MN$.

·10. $AB = AC$ and the $\angle BAC$ is bisected by AX. Any point D is taken on AX. (Fig. (1).)

Prove that $BD = CD$.

11. $ABCDE$ is a regular pentagon (i.e. all its sides and angles are equal. (Fig. (2).)

Prove that BED is an isosceles triangle.

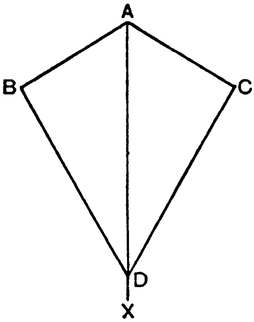


Fig. (1)

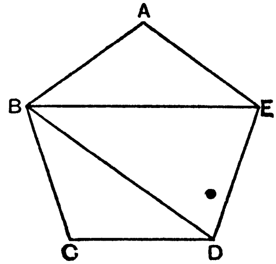


Fig. (2)

12. $ABCD$ is a rectangle (and therefore its opposite sides are equal, and its angles are right angles). (Fig. (3).)

Prove that $AC = BD$.

Note. Imagine the triangles ABC, DCB taken away from the figure and drawn separately.

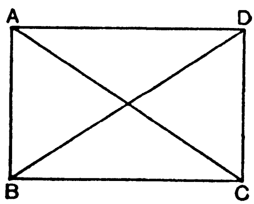


Fig. (3)

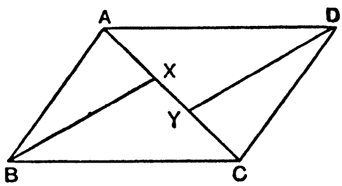


Fig. (4)

13. $ABCD$ is a parallelogram (opposite sides equal) and equal distances AX, CY are cut off on its diagonal. (Fig. (4).)

Prove that $BX = DY$.

14. $ABCDE$ is a regular pentagon. X is the mid-point of AB and Y the mid-point of ED. (Fig. (5).)

Prove that $CX = CY$.

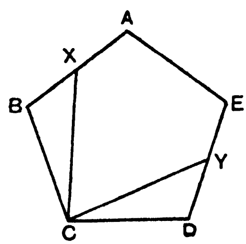
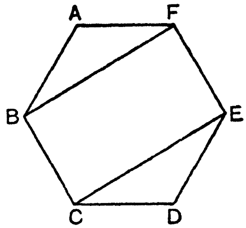


Fig. (5)

15. *ABCDEF* is a regular hexagon.

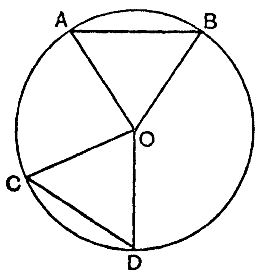


Prove that $BF = CE$.

16. *ABC* is a triangle having its sides AB, AC equal. D, E, F are the middle points of the sides BC, CA, AB respectively. Prove that $DE = DF$.

17. Prove that the triangle formed by joining the mid-points of the sides of an isosceles triangle is also isosceles.

18. AB and CD are two chords of a circle and O the centre, and $\angle AOB = \angle COD$.

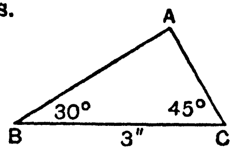


Prove that the chords must be equal.

Note that AOB is said to be the angle *subtended* by AB at the centre.

25. 2nd Case of Congruence of Triangles.

Suppose we had to construct accurately the triangle of which the rough figure is given.



Notice that if we know two angles, we also know the third, for

$$A + B + C = 180°. \qquad \therefore A = 105°.$$

First make BC 3″. Then make an angle of 30° at B and 45° at C.

Here again if a whole class of scholars made accurately a triangle with one side and the angles adjacent to it of given

dimensions, all the triangles would fit together. It might be necessary to turn some of the triangles over, or to turn them round, but there could not be triangles of different dimensions satisfying the given conditions.

In Fig. (1) are shown two triangles with *two angles equal in each* (and therefore three) *and with one side equal in each.* Are they congruent? Yes, because it would be possible to cut out

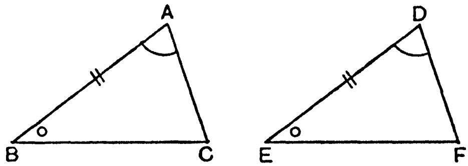


Fig. (1)

DEF and fit it exactly on ABC, for DE and AB would fit together, and the arms of the angles would fit also.

It is not so easy to see that the triangles in Fig. (2) would fit together.

But if DFE were cut out and turned over, it would take the same shape as DEF in Fig. (1). Hence these triangles also are congruent.

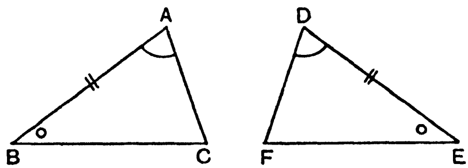


Fig. (2)

All pairs of triangles which have one side and the two angles adjacent to that side in one equal to the corresponding features in the other are congruent.

Are the triangles in Fig. (3) congruent? To find out we must first turn DEF over, when it takes the form shown in Fig. (4). (Both diagrams are purposely inaccurate.)

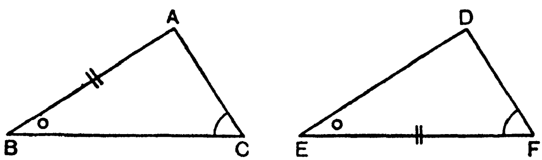


Fig. (3)

The third angles are also equal; that is $\angle D = \angle A$. If we now attempt to place DEF on ABC, we may make FE fit on AB, and the arm ED will be in the direction BC. *But* FD may not be in the direction AC because $\angle F$ is not given equal to $\angle A$.

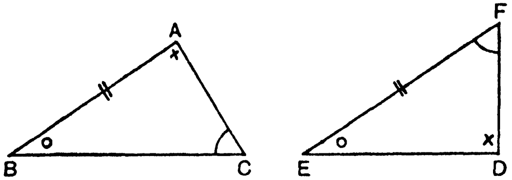
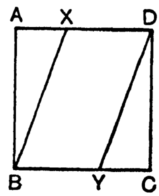


Fig. (4)

The point to notice is that the sides equal in each must be *opposite* to angles which are known to be equal in each. These are called *corresponding* sides.

Learn:

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.



This result is used in the following exercises.

Ex. 1. $ABCD$ is a square.

BX, DY are drawn so that $\angle ABX = \angle CDY$.

Prove that $BX = DY$.

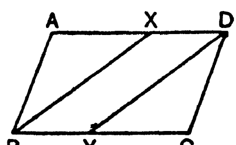
Proof. In the Δs ABX, CDY,

$$\begin{cases} \angle BAX = \angle DCY \text{.......................... rt. } \angle \text{s,} \\ \angle ABX = \angle CDY \text{.......................... given,} \\ AB = CD \text{............................ sides of a sq.} \end{cases}$$

I.e. Two angles and a corresponding side are equal in each.

$\therefore$ the Δs are congruent,

$\therefore BX = DY$.



Ex. 2. $ABCD$ is a parallelogram, and BX, DY bisect the angles at B and D. Assuming that the opposite sides and angles of a parallelogram are equal, prove that $BX = DY$.

Proof. In the $\triangle$s ABX, CDY,

$$\begin{cases} \angle BAX = \angle DCY \text{ ....opp. } \angle\text{s of a parm.,} \\ \angle ABX = \angle CDY \text{ ....halves of opp. } \angle\text{s,} \\ AB = CD \text{ ....opp. sides of a parm.} \end{cases}$$

$\therefore$ the $\triangle$s are congruent and $BX = DY$.

EXERCISE 13.

1. Make the triangle of which the rough figure (Fig. (1)) is shown.

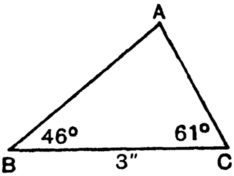


Fig. (1)

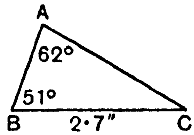


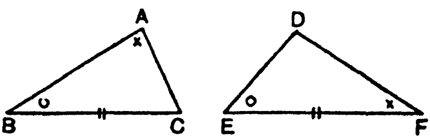
Fig. (2)

2. Calculate $\angle ACB$ and then construct the triangle ABC (Fig. (2)).

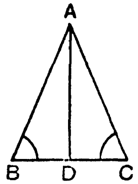
Construct triangles, having given the following parts:

3. $a = 3{\cdot}1''$,	$A = 49°$,	$B = 52°$.
4. $a = 2{\cdot}9''$,	$A = 59°$,	$C = 40°$.
5. $a = 8{\cdot}3$ cm.,	$B = 82°$,	$C = 31°$.
6. $b = 6{\cdot}9$ cm.,	$A = 29°$,	$B = 48°$.
7. $b = 3{\cdot}6''$,	$B = 33°$,	$C = 89°$.
8. $c = 2{\cdot}3''$,	$C = 103°$,	$A = 41°$.
9. $c = 2{\cdot}8''$,	$A = 22°$,	$B = 127°$.
10. $c = 2\frac{3}{4}''$,	$A = C = 50°$.	

11. Are the triangles indicated in the rough figure congruent? Give reasons. (The diagrams are purposely inaccurate.)

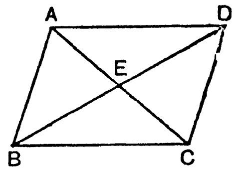


12. The $\triangle ABC$ has $\angle B = \angle C$, and the vertical angle is bisected by AD.



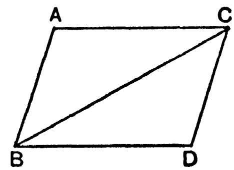
Prove that $AB = AC$.

13. $ABCD$ is a parallelogram. Suppose we know that its opposite sides are equal as well as parallel.

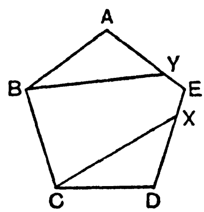


Prove that the diagonals bisect one another (i.e. prove the triangles AED, CEB congruent).

14. Two $\triangle$s ABC, DCB with $BA = CD$, $AC = BD$ and the $\angle BAC = \angle BDC$, are fitted together as shown. Prove that $ABDC$ is a parallelogram (i.e. prove that the opposite sides are parallel).



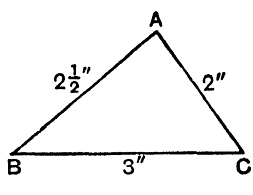
15. $ABCDE$ is a regular pentagon.



CX is drawn so that $\angle XCD = 30°$ and BY so that $\angle ABY = 30°$. Prove that $CX = BY$.

26. 3rd Case of Congruence of Triangles.

You have learnt that, to construct a triangle, given the lengths of the three sides, you first construct one side, e.g. BC; then with centre B and radius $2\frac{1}{2}''$, and with centre C and radius $2''$, describe arcs cutting at A.



You have discovered that no triangle can be constructed if one side is equal to or greater than the sum of the other two sides. It will probably be clear that a triangle of only one size and shape can be constructed with the dimensions given above.

Hence it seems as if two triangles with three sides equal in each must be congruent. Suppose we are merely told that the base is 3″ and the other two sides $2\frac{1}{2}$″ and 2″. Let us try to construct two different triangles.

In Fig. (1) we are assuming that BA must be $2\frac{1}{2}$″ and CA 2″. With centre B and radius $2\frac{1}{2}$″ make a circle.

Every point $2\frac{1}{2}$″ from B must be on the circumference of that circle.

With centre C and radius 2″ make another circle.

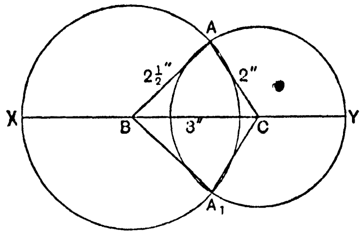


Fig. (1)

Every point 2″ from C must be on the circumference of that circle. A and A_1 are the only points on both circles. $\therefore ABC$ and A_1BC are the only two triangles that can be made on the base BC with BA and AC $2\frac{1}{2}$″ and 2″.

It is easy to see that the bottom part of the figure can be folded over the diameters, which lie on $XBCY$, so that it exactly coincides with the top part.

Hence $\triangle\ A_1BC$ must be congruent with $\triangle ABC$.

But we might have made AB 2″ and AC $2\frac{1}{2}$″, as in Fig. (2).

It seems at first as if we may have found a triangle not congruent with the first.

But if Fig. (2) is turned over it can be made to coincide exactly with Fig. (1). We have therefore proved that two triangles with three sides equal in each must be congruent.

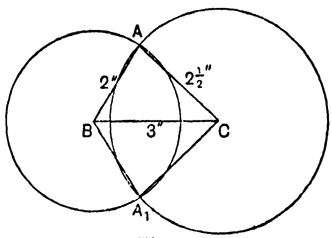


Fig. (2)

Learn:

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.

EXERCISE 14.

Make triangles with sides as follows:

1. $a = 3''$, $b = 4''$, $c = 5''$.
2. $a = 2{\cdot}8''$, $b = 3{\cdot}1''$, $c = 4{\cdot}9''$.
3. $a = 8{\cdot}9$ cm., $b = 5{\cdot}8$ cm., $c = 6{\cdot}3$ cm.
4. $a = 10{\cdot}8$ cm., $b = 6{\cdot}4$ cm., $c = 7{\cdot}9$ cm.
5. $a = b = c = 2{\cdot}3''$.
6. $a = b = 2{\cdot}9''$, $c = 3{\cdot}2''$.
7. $a = 2b = 2c$, $c = 1\frac{1}{2}''$.
8. $a = b = 1\frac{1}{2}c$, $a + b + c = 9''$.
9. $a = 8{\cdot}3$ cm., $b = 4{\cdot}2$ cm., $c = 3{\cdot}6$ cm. What do you notice?
10. $a = 3''$, $b = 1\frac{1}{4}''$, $c = 1\frac{3}{4}''$. What do you notice?

11. These triangles have three sides of one equal to three sides of the other (Fig. (1)). Hence they are congruent. Name the pairs of equal angles.

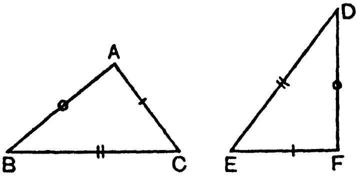


Fig. (1)

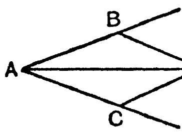


Fig. (2)

12. In bisecting an angle BAC, we arrange that $AB = AC$, and $BD = CD$ (Fig. (2)).

Prove that the angle is bisected.

13. $ABCD$ is a rhombus, i.e. all its sides are equal (Fig. (3)).

Prove that the diagonal AC bisects two opposite angles.

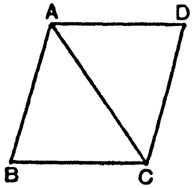


Fig. (3)

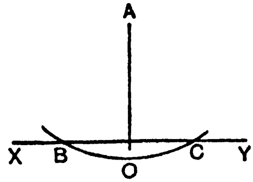


Fig. (4)

14. A is the centre of a circle which cuts a line XY at B and C. BC is bisected at O (Fig. (4)).

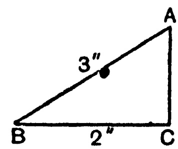
Prove that AO is perpendicular to XY.

15. ABC is a triangle in which $AB = AC$. The bisector of the angle ABC meets AC in E, and the bisector of the angle ACB meets AB in F. Prove that $BE = CF$.

16. Show that two equal chords of a circle subtend equal angles at the centre (i.e. if O is the centre and AB, CD the equal chords, prove $\angle AOB = \angle COD$).

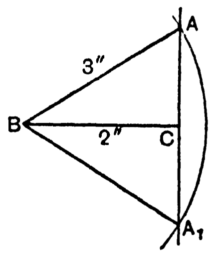
27. A Fourth Case of Congruence of Triangles.

Ex. 1. To construct a right-angled triangle having given the length of the hypotenuse and that of one side.



Suppose ABC is our rough figure, and that the hypotenuse is to be 3″ and BC 2″.

We start by making BC 2″ and raising a perpendicular at C.



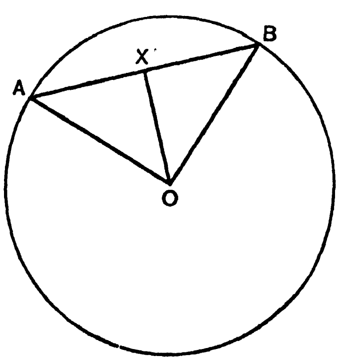
BA has to be made 3″ long, so with centre B and radius 3″ we make an arc of a circle cutting the perpendicular at A and A_1. It cannot cut the perpendicular at any other points.

Also it is obvious that these two triangles are congruent.

Learn:

If two right-angled triangles have the hypotenuse and one side equal in each, the triangles are congruent.

Ex. 2. AB is a chord of a circle and OX is drawn from the centre of the circle perpendicular to AB.



Prove that OX bisects AB.

The following is the statement required:

Construction. Join OA, OB.

Proof. In the right-angled triangles OXA, OXB,

$$\begin{cases} OA = OB, \text{ radii,} \\ OX \text{ is common to both triangles.} \end{cases}$$

$\therefore$ the Δs are congruent and $AX = XB$.

The method of statement is important. We *must* say the triangles are right angled, but we *must not* state the proof as follows:

In the $\triangle$s OXA, OXB,

$$\begin{cases} \angle OXA = \angle OXB, \\ OA = OB, \\ OX \text{ is common,} \end{cases}$$

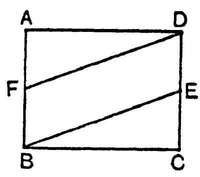
$\therefore$ the triangles are congruent,

or we shall be saying that two triangles are congruent because they have two sides and one angle (*not contained by the two sides*) equal in each.

EXERCISE 15.

Make right-angled triangles according to the following requirements:

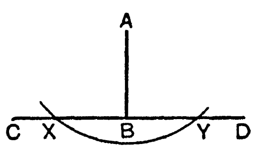
1. $A = 90°$,	$a = 8{\cdot}5$ cm.,	$b = 5{\cdot}1$ cm.
2. $C = 90°$,	$a = 3''$,	$b = 1{\cdot}8''$.
3. $B = 90°$,	$a = 2{\cdot}8''$,	$b = 3{\cdot}2''$.
4. $A = 90°$,	$b = 3{\cdot}7''$,	$c = 2{\cdot}7''$.
5. $B = 90°$,	$b = 4''$,	$c = 2''$.



6. $ABCD$ is a rectangle, and equal lines BE, DF are drawn to cut CD at E and AB at F. Prove that $AF = CE$ and hence that $BF = DE$.

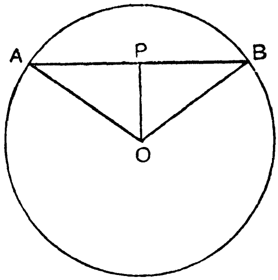
7. $ABCD$ is a rectangle. E, F are points on CD such that $BE = AF$. Prove that $CE = DF$.

8 AB is perpendicular to CD, and with centre A and any radius a circle is drawn cutting CD at X and Y. Prove that $BX = BY$.



9. Assuming that the diagonals of a rhombus bisect the opposite angles, prove that the diagonals bisect at right angles.

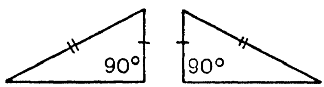
10. From O the centre of a circle a line OP is drawn bisecting AB. Prove that OP is at right angles to AB.



11. Using the previous figure: from O the centre of a circle a line OP is drawn at right angles to the chord. Prove that $AP = PB$.

12. Prove that the line drawn from the vertex of an isosceles triangle perpendicular to the base bisects it.

13. Are the triangles indicated in the rough figure congruent and why?



28. A Case of Conditional Congruence.

Suppose we have to copy the triangle ABC (Fig. (1)), given two sides and *the angle opposite to one of them.*

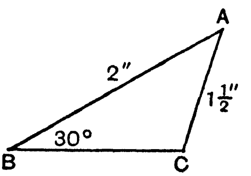


Fig. (1)

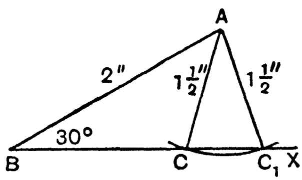


Fig. (2)

We can easily make $\angle ABC$ and cut off BA 2″ on one of the arms (Fig. (2)). As yet we do not know where the point C is situated, but we know that it is $1\frac{1}{2}''$ from A.

Hence with centre A and radius $1\frac{1}{2}''$ we make an arc cutting BX. In this case it cuts BX at two points C and C_1. We therefore have two triangles ABC, ABC_1 satisfying the given conditions.

This is a case in which two triangles have two sides and an angle equal in each, but the triangles are not congruent. The angle in question is not *contained* by the sides equal in each.

If AC happened to be 1″, we should find our arc just touching BX at one point C, and only one triangle would be possible.

If AC were more than 2″, we should find that the arc with centre A and radius 2″ cuts BX at two points, but one point would be to the left of B and this triangle would not have an angle B of 30°. There would only be one possible triangle in this case also.

Finally if AC were less than 1″, the arc would neither touch nor cut BX. No triangle would be possible.

We therefore have to consider four cases:

(1) Where the arc touches BX...1 solution.

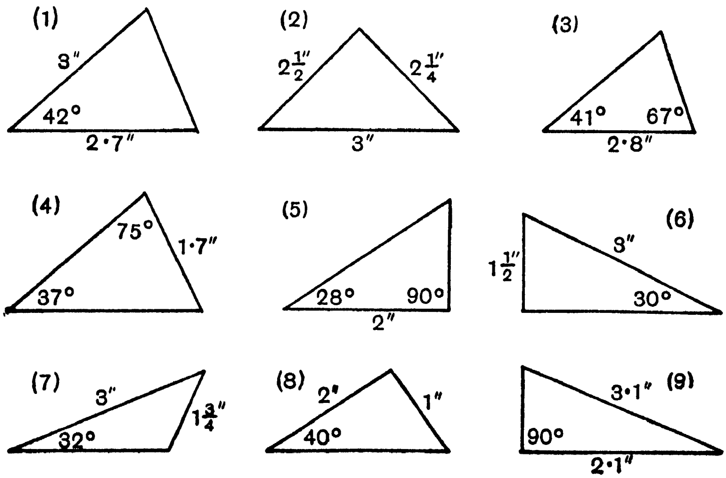
(2) Where the arc cuts BX at two points, one to the left of B...1 solution.

(3) Where the arc cuts BX at two points to the right of B... 2 solutions.

(4) Where the arc neither touches nor cuts BX...no solution.

EXERCISE 16.

Construct triangles, where possible, with the dimensions indicated in the rough diagrams:



Construct triangles, where possible, with the following dimensions:

10. $A = 50°$, $B = 60°$, $a = 2{\cdot}3''$.
11. $A = 43°$, $b = 1{\cdot}9''$, $c = 2{\cdot}4''$.
12. $a = 12{\cdot}3$ cm., $b = 11{\cdot}9$ cm., $c = 10{\cdot}3$ cm.
13. $A = 102°$, $B = 38°$, $c = 1{\cdot}1''$.
14. $a = 2{\cdot}8''$, $b = 1{\cdot}8''$, $A = 55°$.
15. $a = 2{\cdot}9''$, $b = 1{\cdot}3''$, $B = 37°$.

16. ABC is a triangle having its sides AB, AC equal. E and F are the middle points of the sides CA and AB respectively. Prove that $BE = CF$.

17. $ABCD$ is a rectangle (i.e. a quadrilateral with its opposite sides equal and parallel and all its angles right angles), and P is any point equidistant from A and B. Prove that $PC = PD$.

18. Draw any triangle ABC. Produce BA to D so that $AD = AC$, and CA to E so that $AE = AB$. Prove that $DE = BC$, and CD is parallel to BE.

19. If in a triangle $AB = AC$, prove that the bisector of the angle A bisects the base BC at right angles.

20. Two circles have the same centre O. Radii OA, OB of the smaller circle are produced to meet the larger in C and D. Prove that CD is parallel to AB.

21. "The parts of a triangle being its 3 angles and its 3 sides, two triangles are *always* congruent when 3 corresponding parts are equal, each to each." State, *without proof*:

(*a*) Three sets of 3 corresponding parts which corroborate this statement;

(*b*) Two sets of 3 corresponding parts which contradict it.

In cases (*b*) indicate with the help of sketches the relations between the triangles.

22. Construct the triangle ABC in which $AB = 9{\cdot}8$ cm., the angle $A = 37°$, and the angle $C = 99°$. Find a point P equidistant from A and B, and also equidistant from AB and AC. Measure AP.

23. If the perpendiculars on two sides of a triangle from the extremities of the base are equal, prove that the given triangle must be isosceles.

24. Two places A and B lie east and west of one another. From A a man goes due south 8 miles and then due west 4 miles. From B another man goes due north 4 miles and then due east 8 miles. Prove that the second is now as far from B as the first is from A.

25. From a given point X draw a straight line XY making an angle of 30° with a given straight line AB and another line XZ making an angle of 55° with it. Measure YZ.

26. An aeroplane A is just in the vertical plane through two ground stations B and C which are 1500 yards apart. If the angle $CBA = 25°$, and the angle $BCA = 48°$, draw the triangle ABC to a scale of 1000 feet = 1 inch, and find the vertical height of the aeroplane above the ground.

27. A man observes two objects at the same time, one exactly behind the other, and both in the north-west direction. He walks 600 yards due west, and then finds that one object is due north and the other north-east. Find how far he was from each object at first.

28. Construct a triangle PQR having $PQ = 6{\cdot}7$ cm., $PR = 2{\cdot}9$ cm., and $RQ = 5{\cdot}6$ cm. Draw a line RS cutting QP produced in S, so that the angle PRS is equal to the angle PQR.

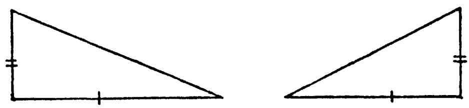
29. A ladder, $17\frac{1}{2}$ feet long, leans against a wall 12 feet high. When the foot of the ladder is in position A on the ground, a length of 3 feet 6 inches of the ladder extends beyond the top of the wall, and when in position B there is a length of 1 foot 6 inches above the wall. Make a scale drawing showing the two positions of the ladder.

30. Five towns A, B, C, D, E are situated such that the distance from B to C is 45 miles, C to D 50 miles, C to A 80 miles, D to B 70 miles, D to E 50 miles, D to A 70 miles, E to A 50 miles. Show the positions of the towns by a scale drawing in which 1″ represents 20 miles.

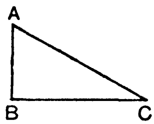
SECTIONAL REVISION B

EXERCISE 17 (*a*). MENTAL.

1. Would these right-angled triangles be congruent if accurately constructed? Give reasons.

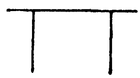


2. *ABC* is a right-angled triangle with $\angle BAC$ 60°. If a second triangle of the same size were made on the other side of *BC*, what figure would be formed?



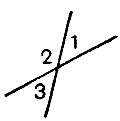
3. One road runs north and south, and another runs north-east and south-west. What is the angle between them?

4. The figure shows two lines drawn perpendicular to another line. Why are they parallel?



5. The angles of a quadrilateral are $2x$, $2x$, $3x$ and $5x$ degrees. What is the value of x?

6. What are angles 1 and 3 called with respect to one another, and what are angles 1 and 2 called?



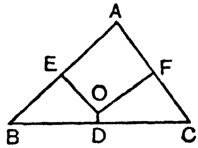
7. Express an angle of $1\frac{3}{5}$ right angles in degrees.

8. The distance from *A* to *B* is 150 miles, from *B* to *C* 120 miles, and from *C* to *A* 270 miles. How do you know that *A*, *B* and *C* are in the same straight line?

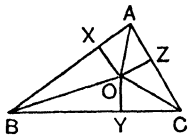
EXERCISE 17 (*b*).

1. Two triangles ABC and DEF have $\angle A = \angle D$. Prove that $\angle B + \angle C = \angle D + \angle F$.

2. Prove that the diagonal of a rectangle divides it into two triangles equal in area.



3. OE, OF bisect AB, AC at right angles. Prove that OD must bisect BC at right angles.

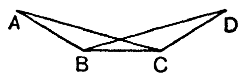


4. OB bisects $\angle ABC$. Prove that $OX = OY$ where these are perpendicular to the sides.

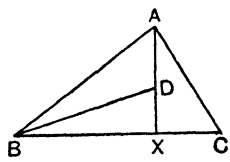
OA bisects $\angle BAC$. Prove similarly that $OX = OZ$.

5. Using the above results, prove that OC bisects $\angle ACB$.

6. What angle is formed at the centre of a circle by radii from the extremities of one side of a regular 12-sided figure described in the circle?



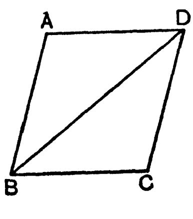
7. AB, BC, CD are three sides of a many-sided regular polygon. Prove that $AC = BD$.



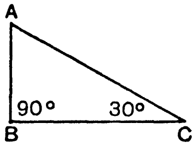
8. If in the figure $\angle ABC$ is bisected by BD and AX is perpendicular to BC, calculate $\angle ADB$ if $\angle ABC = 40°$.

EXERCISE 18 (*a*). MENTAL.

1. $ABCD$ is a rhombus. How do we know that ADB, CDB are congruent triangles?



2. If $\frac{AB}{AC}$ is always the same for an angle of 30°, whatever the lengths of AB, AC, find AB if $AC = 5{\cdot}9$ cm.



3. If I turn from a direction 10° east of north to a direction 20° west of north, through what angle have I turned?

4. How many pairs of parallel edges are there in all the faces of a cube?

5. If one angle of a triangle is $\frac{2}{4}$ of a right angle and another is a right angle, how many degrees are there in the third angle?

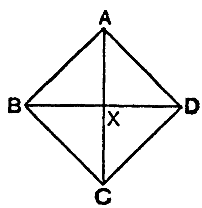
6. The angles of regular polygons gradually increase in size as the number of sides is increased. What is the value of the angle which is not quite reached, when the number of sides is infinitely great?

7. How many corners has a regular prism on a base of 12 sides?

8. If in a triangle ABC, $\angle B = \angle C$, which two sides are equal?

EXERCISE 18 (*b*).

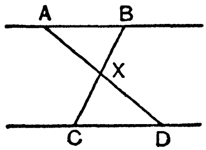
1. $ABCD$ is a rhombus. Prove that $\angle BAC = \angle CAD$.



2. Hence prove that Δs AXB, AXD are congruent and therefore $BX = XD$.

3. ABC is an isosceles triangle and XY is drawn parallel to the base BC to cut the sides AB, AC at X and Y. Prove that AXY is an isosceles triangle.

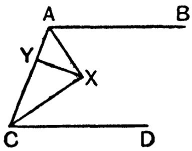
4. AB and CD are parallel lines and AD, BC are any two lines cutting at X. Prove that the triangles AXB, CXD are equiangular.



5. Construct a triangle of which the longest side is 2·9″, the angles being such that the largest is three times the smallest and the other twice the smallest.

6. Construct a parallelogram having diagonals 8·2 cm. and 5·9 cm.

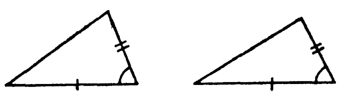
7. Angles BAC and ACD are bisected by AX and CX, and XY is perpendicular to AC. If AB and CD are parallel and $\angle CAB = 124°$, calculate $\angle AXY$.



8. Two sides of an isosceles triangle are 3·2″ each and one of the base angles is 70°. Construct the triangle and draw its altitude.

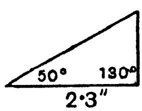
EXERCISE 19 (*a*). Mental.

1. If the triangles indicated in the rough figures were drawn accurately, why would they be congruent?



2. What is the ratio of the smallest side of a triangle with two angles of 30° and 60° to the longest side?

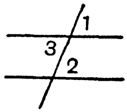
3. Why is it impossible to construct a triangle with these dimensions?



4. If the sides of an isosceles triangle with base angles of 60° each are produced, what is the size of each angle on the other side of the base?

5. Two lines cut at an angle of 87°. At what other angle might they be said to cut?

6. What are angles 1 and 2 called with respect to one another, and what are angles 2 and 3 called?



7. Express an angle of $112\frac{1}{2}°$ as a fraction of a right angle.

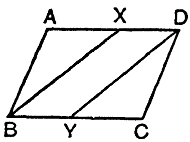
8. What is the perimeter of a square if each edge is 5·85 cm. long?

EXERCISE 19 (*b*).

1. Prove that any point on the perpendicular bisector of a given line is equidistant from the two ends.

2. Two straight lines AB, CD cut at X. If each is bisected at X, prove that $AC = BD$.

3. $ABCD$ is a parallelogram and the angles at B and D are bisected by BX, DY. Prove that $BX = DY$.



4. If the opposite angles of a parallelogram are together equal to two right angles, prove that the figure must be a rectangle.

5. Calculate the angle subtended at the centre of a circle by one side of a regular five-sided figure inscribed in it. Hence construct a regular pentagon in a circle of 1·8″ radius.

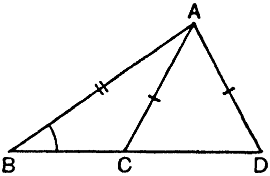
6. Construct a parallelogram with two sides 8·5 cm. long, 4·8 cm. apart, and with one angle of 108°.

7. How many triangles can be constructed with two sides 3·3″ and 1·1″ and an angle of 36° opposite the shorter of these sides?

8. Bisect an angle and prove by means of congruent triangles that you have bisected it.

EXERCISE 20 (*a*). MENTAL.

1. ABC, ABD are two triangles with two sides and one angle equal in each. Why is $\angle ACB$ supplementary to $\angle ADB$?

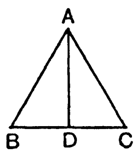


2. What name is given to the triangle ACD in the above figure?

3. What is the angle between the hands of a clock at 3.30 o'clock?

4. How many pairs of parallel edges are there in the faces of a hexagonal prism?

5. ABC is an equilateral triangle and AD bisects the vertical angle. Make a rough copy and mark on it the equal parts which prove Δs ADB, ADC congruent.

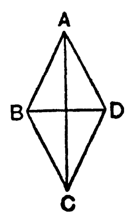


6. I go 5·27 miles N., then 2·3 miles E., then 1·7 miles S., and finally 1·1 miles W. How many miles am I then N. and E. of my starting-point?

7. How many edges has a regular pyramid on a base of 10 sides?

8. What do you understand by the term congruent triangles?

EXERCISE 20 (*b*).



1. ABD and BCD are two equal isosceles triangles. Prove that Δs ABC, ADC are congruent.

2. Hence prove that BD bisects AC at right angles.

3. Construct a quadrilateral $ABCD$ in which $AB = BC = 2{\cdot}5''$, $AD = CD = 1{\cdot}8''$, and $\angle ABC = 90°$.

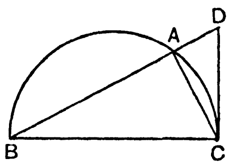
4. Using the figure of Question 3, prove that $\angle ADB = \angle BDC$.

5. Describe two triangles which are not congruent, having one side $3{\cdot}8''$, another $2{\cdot}7''$, and the angle opposite to the latter $28°$.

6. What angle is subtended at the centre of a circle by one side of a regular hexagon described in it? Hence construct such a figure on a base of $1{\cdot}7$ inches.

7. Prove that the perpendicular from the centre of a circle to a chord bisects the chord.

8. Construct this figure if $BC = 3{\cdot}9''$, BAC is a semicircle and DC at right angles to BC.



AREAS OF TRIANGLES AND QUADRILATERALS

29. Some Kinds of Quadrilaterals.

Parallelogram.

Opposite sides parallel.

Trapezium or Trapezoid.

One pair only of parallel sides.

Rectangle or Oblong.

Parallelogram with one angle a right angle. *Hence all its angles are right angles.*

Square.

Rectangle with two adjacent sides equal. *Hence all its sides are equal.*

Rhombus.

All sides equal but none of its angles a right angle.

30. Construction of Quadrilaterals.

Ex. 1. Here we make CD 2″, then $\angle$s BCD and ADC (Fig. (1)).
BC is made 1″, AD 2″.
Finally A, B are joined.

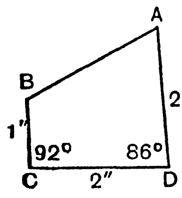


Fig. (1)

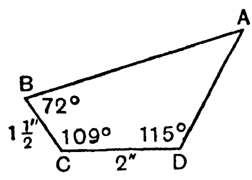
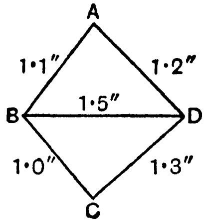


Fig. (2)

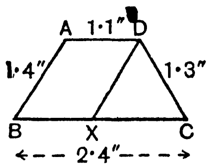
Ex. 2. BC, CD and $\angle BCD$ are made first (Fig. (2)).

Next we construct $\angle$s at B and D, and the arms BA, DA meet at A.

Ex. 3. When the four sides and the length of a diagonal are given we first make one of the two triangles, and then the other on the diagonal.



Ex. 4. To construct a trapezium given the lengths of the four sides, with an indication as to which sides are parallel.



In the rough figure draw DX parallel to AB.

$\therefore ABXD$ is a parallelogram, and $DX = 1{\cdot}4''$.

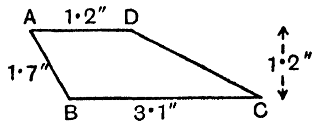
BX now equals $1{\cdot}1''$. $\therefore XC$ is $1{\cdot}3''$.

First construct $\triangle DXC$ with sides $1{\cdot}4''$, $1{\cdot}3''$, $1{\cdot}3''$.

Produce CX to B making CB $2{\cdot}4''$. Draw DA parallel to BC and $1{\cdot}1''$ and join AB. Then $ABCD$ is the trapezium required.

Ex. 5. To construct a trapezium given the lengths and the distance apart of the parallel lines, and the length of one side.

Make BC $3{\cdot}1''$.



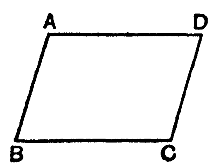
Draw a line parallel to BC and $1{\cdot}2''$ away.

With centre B and radius $1{\cdot}7''$ cut this parallel at A.

Make AD $1{\cdot}2''$ long. Then $ABCD$ is the trapezium.

Notice that the circle with centre B and radius $1{\cdot}7''$ cuts the parallel to BC at two points. Hence there are two solutions of the problem.

Ex. 6. To construct a parallelogram given two sides and the angle contained by them.

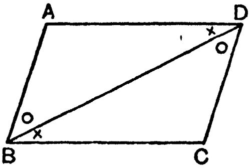


The $\angle ABC$, AB and BC are drawn to given dimensions; AD is then drawn parallel to BC and CD parallel to BA; or with centre A and radius equal to BC, and with centre C and radius equal to BA, arcs are drawn cutting at D.

31. Parallelograms.

The definition of a parallelogram merely states that it is a four-sided figure with its opposite sides parallel.

But we can also prove that the opposite sides and angles are equal, and that either diagonal divides it into two exactly equal portions.



Proof. ABD and DBC are congruent because

$$\begin{cases}\text{Alternate } \angle\text{s } ADB,\ DBC \text{ are equal,}\\ \text{Alternate } \angle\text{s } ABD,\ CDB \text{ are equal,}\\ BD \text{ is common to both triangles.}\end{cases}$$

Hence (1) $AB = CD$,

(2) $AD = CB$,

(3) $\angle BAD = \angle DCB$,

and since $\angle ADB = \angle DBC$, and $\angle CDB = \angle ABD$,

$\therefore$ (4) $\angle ADC = \angle CBA$.

Learn:

The opposite sides of a parallelogram are equal.

The opposite angles of a parallelogram are equal.

Each diagonal of a parallelogram divides the figure into two equal triangles.

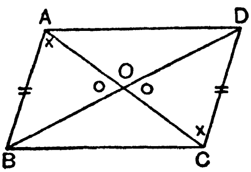
32. The diagonals of a parallelogram bisect one another.

The Δs ABO, CDO are congruent because

$$\begin{cases}\angle AOB = \angle COD. \text{ Vertically opposite.}\\ \angle BAO = \angle DCO. \text{ Alternate.}\\ AB = CD, \text{ as proved above.}\end{cases}$$

$\therefore AO = OC,$

$BO = DO.$



Learn:

The diagonals of a parallelogram bisect one another.

33. Construction of Parallelograms and Trapezoids.

The construction of these figures to given dimensions will become simple if the following principles are remembered:

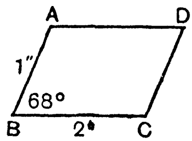
1. *A rough figure should always be drawn, and the given dimensions marked upon it.*
2. *A parallelogram has two pairs of parallel sides, and a trapezoid one pair.*
3. *The diagonals of a parallelogram bisect one another.*

Consider the following hints for construction, the rough figures indicating the dimensions.

Parallelograms.

1. Here we copy AB, BC and $\angle ABC$.

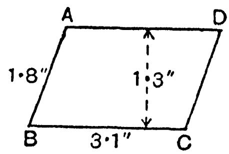
Then AD is drawn parallel to BC, and CD parallel to BA.



2. The parallels are drawn $1{\cdot}3''$ apart.

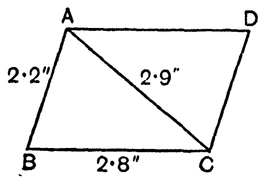
BC is made $3{\cdot}1''$.

With centre B and radius $1{\cdot}8''$ an arc is drawn cutting the parallel AD.



3. Here we make the triangle ABC, of which the three sides are given.

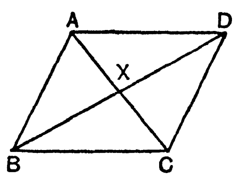
The rest is obvious.



4. Given diagonals $3''$, $2''$ and BC $1{\cdot}7''$.

The diagonals bisect. $\therefore BX = 1{\cdot}5''$, $CX = 1''$.

Make BXC with the three sides $1{\cdot}7''$, $1{\cdot}5''$, $1{\cdot}0''$.

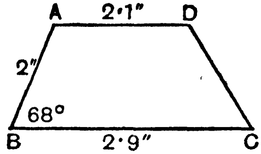


Produce the half-diagonals so as to complete them, and join AB, AD, CD.

Trapezoids.

1. AB, BC and $\angle ABC$ are made first.

Then AD is drawn parallel to BC and made 2·1″.

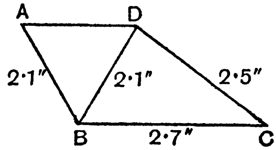


2. The triangle DBC is made first.

DA is drawn parallel to BC.

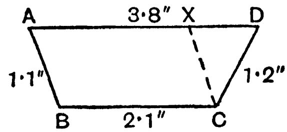
Then BA is made 2·1″.

Note that two different figures are sometimes possible.



3. If the four sides are given, we divide the figure into a parallelogram and a triangle.

First construct the triangle with sides 1·2″, 1·1″, and 3·8″—2·1″.



EXERCISE 21.

1. $ABCD$ is a parallelogram (Fig. (1).) Write down four pairs of equalities.

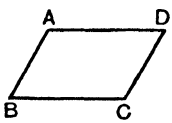


Fig. (1)

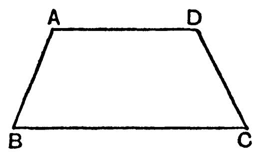


Fig. (2)

2. $ABCD$ is a trapezoid (Fig. (2).) Show how you would divide it into a parallelogram and a triangle.

Construct parallelograms having:

3. Adjacent sides 2·3″, 1·2″ and one angle 62°.

4. Adjacent sides 8·8 cm., 6·2 cm. and one angle 98°.

5. Adjacent sides 3·7″, 2·2″, with the longer sides 1·3″ apart.

6. Adjacent sides 8·3 cm., 10·8 cm., with the shorter sides 7·9 cm. apart.

7. Adjacent sides 3·1″, 3·2″ and a diagonal 4·5″.

8. Adjacent sides 9·2 cm., 9·2 cm. and a diagonal 9·2 cm.

9. Diagonals $2{\cdot}9''$, $3{\cdot}3''$, containing an angle of $123°$.

10. Diagonals $7{\cdot}5$ cm., $7{\cdot}2$ cm., containing an angle of $52°$.

11.* Diagonals $3{\cdot}1''$, $3{\cdot}0''$ and one side $4{\cdot}0''$.

Construct trapezoids $ABCD$ having AD, BC parallel and:

12. AB $2''$, BC $3{\cdot}8''$, BD $3{\cdot}5''$, AD $2''$.

13. AB $2{\cdot}6''$, BC $3{\cdot}5''$, AD $1{\cdot}1''$, $\angle ABC$ $60°$.

14. AB $1{\cdot}7''$, BC $3{\cdot}4''$, CD $1{\cdot}9''$, AD $1{\cdot}5''$.

15. BC $2{\cdot}8''$, CD $1{\cdot}3''$, AD $1{\cdot}1''$, $\angle BCD$ $35°$.

Construct quadrilaterals $ABCD$ with the following dimensions:

16. $AB = 1{\cdot}8''$, $BC = 2{\cdot}9''$, $CD = 1{\cdot}5''$, $\angle B = 60°$, $\angle C = 85°$.

17. $AB = 5{\cdot}7$ cm., $CD = 5{\cdot}9$ cm., $AD = 8{\cdot}2$ cm., $\angle A = 71°$, $\angle D = 58°$.

18. $AB = 2''$, $BC = 2''$, $CD = 3''$, $DA = 3''$, $BD = 3''$.

19. $AB = 4{\cdot}1$ cm., $BC = 4{\cdot}0$ cm., $CD = 5{\cdot}5$ cm., $DA = 5{\cdot}6$ cm., $\angle A = 85°$.

20. Construct a quadrilateral $ABCD$ with

$$AB = AC = AD = CD = 2 \text{ in.},$$

and the angle $BAC = 30°$.

(i) Measure, (ii) calculate the angles ABC, ABD.

If AC and BD intersect at E, prove that $BC = BE$.

21. Construct a parallelogram in which the diagonals are $6''$ and $4''$ and the shorter sides are each $3''$. State your method, and measure the longer sides.

22. Construct a quadrilateral $ABCD$ in which

$$AB = CD = 2'', \quad BC = 2{\cdot}5'',$$

the diagonal $AC = 4''$, and the angle $CAD = 30°$.

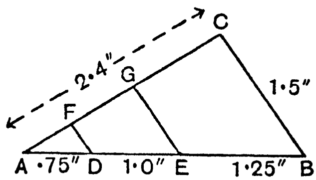
23. Construct a rhombus $ABCD$ in which the angle $ADC = 120°$ and the diagonal BD is $2{\cdot}5$ inches long. If P is a point on BD produced such that $PD = DB$, prove that PAB is a right angle.

24. Construct a pentagon $ABCDE$, having $AB = 5{\cdot}0$ cm., $BC = 4{\cdot}1$ cm., $CD = 3{\cdot}6$ cm., $DE = 7{\cdot}4$ cm., and $EA = 4{\cdot}5$ cm.; also angle $ABC = 105°$, and the angle $EAB = 143°$.

25. A man walks $4\frac{1}{2}$ miles to the north, then $3\frac{1}{4}$ miles south-east, then 5 miles south, and finally 2 miles north-west. Represent this journey by a drawing to a scale of $\frac{3}{4}$ inch to a mile. How far is the man from his starting-point at the end of the journey?

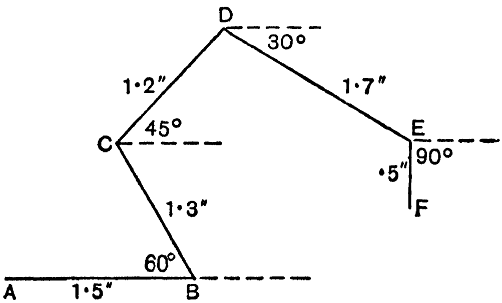
26. Construct geometrically the triangle ABC to the given dimensions, and by using set-squares draw DF, EG parallel to BC.

Measure the lengths of AF, FG, GC and compare $\frac{AF}{FG}$, $\frac{FG}{GC}$ with $\frac{AD}{DE}$, $\frac{DE}{EB}$.



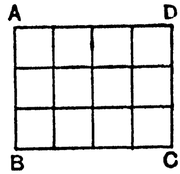
27. $ABCD$ is a quadrilateral in which $AB = 2{\cdot}9''$, $BC = 3{\cdot}1''$, $CD = 3{\cdot}5''$, $DA = 2{\cdot}5''$, and $AC = 3{\cdot}8''$. Bisect the angles at B and D and find the length of the section of AC between the intersections with it of the bisectors.

28. Starting from the point A, draw the figure $ABCDEF$ to the given dimensions. Measure the length of the line AF and the magnitudes of the angles BAF and CDE.



AREAS

34. The Area of a Rectangle.



Suppose the rectangle $ABCD$ has sides AD 4″, and AB 3″.

Let AD be divided into four equal parts of 1″ each, and AB into three of 1″ each.

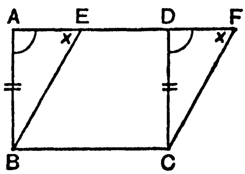
Draw lines parallel to the sides through the points of division, thus dividing the figure into 12 equal squares of 1″ side.

It is seen from the figure that the area is 12 square inches. Similarly, if the sides are $2\frac{1}{2}$ inches and $2\frac{1}{4}$ inches, the area is $2\frac{1}{2} \times 2\frac{1}{4}$ square inches, for the figure can be divided into 10×9, i.e. 90 quarter-inch squares, and

$$2\tfrac{1}{2}'' \times 2\tfrac{1}{4}'' = \tfrac{5}{2}'' \times \tfrac{9}{4}'' = \tfrac{45}{8} \text{ or } \tfrac{90}{16} \text{ square inches.}$$

In general the area of a rectangle = base × height, or $A = bh$.

The Area of a Parallelogram.



Let $EBCF$ be a parallelogram on the same base as the rectangle $ABCD$ and of the same height.

The triangles ABE, DCF are congruent, for the reasons indicated in the diagram.

If we add the trapezoid $EBCD$ to each of these, we see that

$$ABCD = EBCF.$$

But $$ABCD = bh,$$

$$\therefore\ EBCF = bh,$$

since both have the same base and the same height.

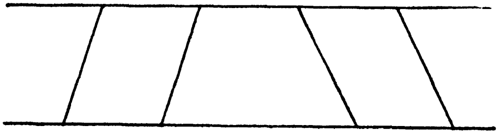
Learn:

The area of a parallelogram is found by multiplying base by height, both being given in the same units; or $A = bh$.

All parallelograms having equal bases and heights are equal in area.

The latter is obvious, since b and h, and therefore $b \times h$, will be the same for all.

Note:



The above parallelograms lie between the same parallels. The heights of the parallelograms must be the same, since parallels are always the same distance apart.

35. The Area of a Triangle.

In Fig. (1) the triangle ABC is half the rectangle $DBCE$ of the same height and on the same base, since

$$\triangle ABX = \tfrac{1}{2} \text{ the rectangle } DX,$$

and $$\triangle ACX = \tfrac{1}{2} \text{ the rectangle } EX.$$

By adding we get

$$ABC = \tfrac{1}{2} \text{ the rectangle } DBCE.$$

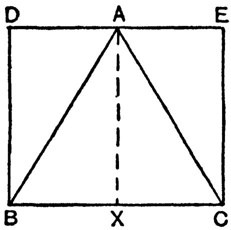


Fig. (1)

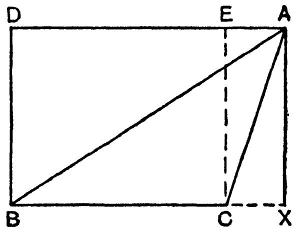


Fig. (2)

In Fig. (2) we must subtract the results, and once again we get

$$\triangle ABC = \tfrac{1}{2} DBCE.$$

But $$DBCE = bh, \quad \therefore ABC = \tfrac{1}{2} bh,$$

since bases and heights of triangle and rectangle are equal.

Learn:

The area of a triangle is half that of a rectangle with the same base and height, or $\triangle = \frac{1}{2} bh$.

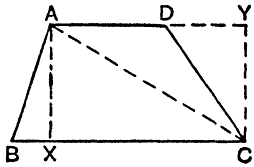
All triangles having equal bases and heights are equal in area.

All equal triangles having equal bases have equal heights.

All equal triangles having equal heights have equal bases.

Note also that triangles lying between the same parallels must have equal heights.

36. The Area of a Trapezium or Trapezoid.



All trapezoids have one pair of parallel sides.

Here AD is parallel to BC.

Divide it into two triangles ABC, ADC by joining AC.

Call BC the base of $\triangle ABC$. Then AX is the height, and $ABC = \frac{1}{2} BC . AX$.

Call AD the base of $\triangle$ ADC. Then CY is the height, and $ADC = \frac{1}{2} AD . CY$.

But $AX = CY$, because these are distances between parallel lines.

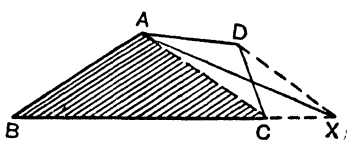
Adding results we get:

$$\begin{aligned} ABCD &= ABC + ADC \\ &= \tfrac{1}{2} BC . AX + \tfrac{1}{2} AD . AX \\ &= \tfrac{1}{2} AX (BC + AD) \\ &= \tfrac{1}{2} \text{(sum of parallel sides)} \times \text{distance between them.} \end{aligned}$$

Learn:

The area of a trapezium is found from the formula $Area = \frac{1}{2}h(a+b)$, *where* a *and* b *are the lengths of the parallel sides and* h *the distance between them.*

37. To make a triangle equal in area to a given quadrilateral.



This is done by joining any two alternate vertices A and C, drawing DX parallel to AC and joining A to X, the point where DX meets BC produced.

Then $ABX = ABCD$.

Proof. ADC, AXC are on the same base and of the same height—because they are between the same parallels.

$$\therefore ADC = AXC.$$

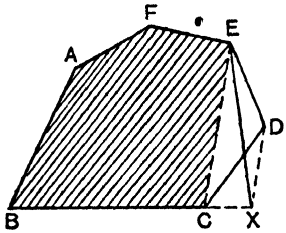
Add the shaded part ABC.

$$\therefore ABC + ADC = ABC + AXC,$$

or

$$ABCD = ABX.$$

38. To reduce any rectilineal figure to a figure of equal area having one side less.



The process is the same as above.

Join any two alternate vertices, e.g. EC.

Draw DX parallel to EC, and join EX as before.

Then $ABXEF = ABCDEF$.

Proof. $EDC = EXC$, as before.

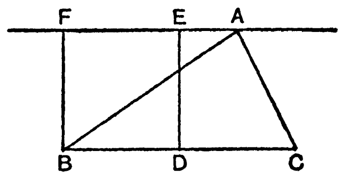
Add the shaded part $ABCEF$.

$$\therefore ABCEF + EDC = ABCEF + EXC,$$

or $$ABCDEF = ABXEF.$$

By continuing the process we can reduce any rectilineal figure to an equal triangle.

39. To make a rectangle equal in area to a given triangle.



Since the area of a $\triangle = \frac{1}{2} b . h$ and a rectangle $= b . h$, there are two simple ways of constructing a rectangle equal in area to a triangle:

(1) Keep the same base and halve the height;

or (2) Keep the same height and halve the base.

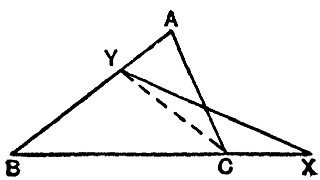
Adopting the latter process, first draw through A a line parallel to BC.

Bisect BC at D.

Draw perpendiculars to BC through B and D.

Then $FBDE$ is a rectangle equal to ABC.

40. To make a triangle on a given base equal to a given triangle.



In the rough figure let ABC represent the given $\triangle$ and BX the base of the new triangle.

Obviously the new $\triangle$ must be of a less height than that of ABC.

Let YBX be the new triangle.

Join YC.

Now if $ABC = YBX$, then $AYC + YBC = YBC + YCX$.

$\therefore AYC = YCX$.

But these are on the same base.

$\therefore$ they are of the same height. $\therefore AX$ is parallel to CY.

The process of working back from this for the construction of the triangle is left as an exercise to the student.

41. The Area of a Triangle by Formula.

The proof of the formula for the area of a triangle, given the lengths of the three sides, is too difficult at this stage, but the formula should be learnt and used where required:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

where s = half the sum of the sides, and a, b, c are the lengths of the sides.

Ex. Find the area of a triangle with sides 7, 8 and 11 inches.

$$s = \frac{7+8+11}{2} = \frac{26}{2} = 13.$$

$$s-a=6, \quad s-b=5, \quad s-c=2.$$

$$\therefore \Delta = \sqrt{13 \times 6 \times 5 \times 2}$$

$$= \sqrt{780} = \text{approximately } 28 \text{ sq. inches.}$$

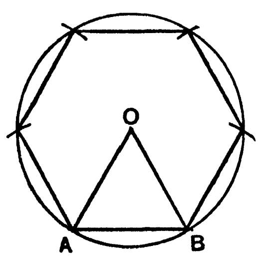
42. Area of any Rectilineal Figure.

(1) *Irregular.*

Divide into triangles and find the sum of the areas of the triangles.

(2) *Regular.*

Every regular polygon can be described in a circle.



The centre is found by bisecting any two sides at right angles.

Let O be the centre of such a polygon.

Join OA, OB.

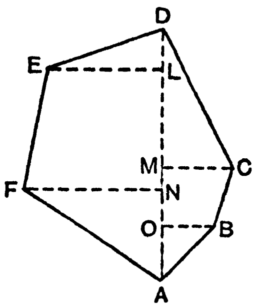
Here there are 6 sides, $\therefore$ 6 triangles equal to OAB will be formed.

Calculate the area of OAB, either by formula as above, or by using the altitude of $\triangle OAB$.

Multiply the result by the number of sides, i.e. by the number of triangles.

Further Examples:

Ex. 1. Find the area of the field illustrated in the given diagram, the offsets and their projections on the base line being of the lengths indicated.



Yards

	D	
	400	
200	300	
	200	120
250	150	
	100	100
	A	

From the Field-Book figures we get:

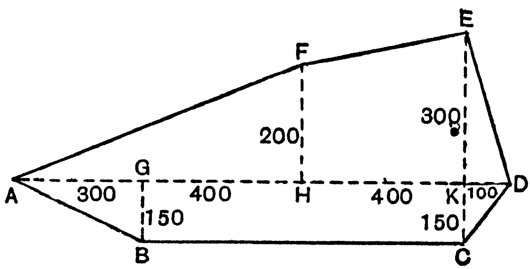
$AO = 100$; $OM = 100$; $AN = 150$; $NL = 150$; $MD = 200$; $LD = 100$.

$$\begin{aligned}
\text{Total area} &= \triangle\, ELD + \text{trapezoid } EFNL + \triangle\, FNA + \triangle\, OAB \\
&\quad + \text{trapezoid } OMCB + \triangle\, DMC \\
&= \tfrac{1}{2}\,.\,200\,.\,100 + \tfrac{1}{2}\,(200+250)\,.\,150 + \tfrac{1}{2}\,.\,250\,.\,150 \\
&\quad + \tfrac{1}{2}\,.\,100\,.\,100 + \tfrac{1}{2}\,(120+100)\,.\,100 + \tfrac{1}{2}\,.\,200\,.\,120 \\
&= 200 \times 50 + 225 \times 150 + 125 \times 150 + 100 \times 50 \\
&\quad + 110 \times 100 + 60 \times 200 \\
&= 200\,(50+60) + 150\,(225+125) + 100\,(50+110) \\
&= 22{,}000 + 52{,}500 + 16{,}000 \\
&= 90{,}500 \text{ sq. yds.} \\
&= \text{approximately } 19 \text{ acres.}
\end{aligned}$$

Ex. 2. Interpret the following Field-Book dimensions, draw a rough diagram of the field, and calculate its area.

The base line is AD, 1200 links, and this is measured from West to East.

	Links	
	D	
	1200	
300	1100	150
200	700	
	300	150
	A	GoE.



The areas can now be calculated as before.

EXERCISE 22.

Find, correct to one place of decimals, the areas of rectangles having sides:

1. 3·72″, 2·3″. 2. 8·72 cm., 7·62 cm.

3. 4·15 ft., 2·75 ft. (sq. ft. and to the nearest sq. inch).

4. 1·035 metres, ·865 metre (sq. cm. to the nearest first place of decimals).

What are the areas of the following parallelograms?

5. Base 5·6″, height 2·9″.

6. Base 8·75 cm., height 5·4 cm.

Find the areas of the following triangles:

7. Base 5·4″, height 2·5″.

8. Base 3·9″, height 7·2″.

What are the areas of the following trapezia?

9. Parallel sides 1·93″, 1·07″; distance apart 2·5″.

10. Parallel sides 5·6 cm., 2·4 cm.; distance apart 5 cm.

11. Construct a parallelogram with two adjacent sides 2·8″, 2·1″, and a contained angle 73°. Find its height by measurement and calculate the area.

12. Find by measurement and calculation the area of a parallelogram with two sides 5·6 cm., 5·2 cm., and a diagonal 6·8 cm.

Draw the following triangles as accurately as you can, and by measuring one altitude calculate the areas:

13. Sides 3·5″, 3·5″, 3·5″.

14. Sides 3·7″, 3·1″, and the angle contained by them 49°.

15. Two angles 43°, 87°, and a side adjacent to these angles 8·75 cm.

16. Two angles 30° 30′, 48° 30′, and the smallest side 1·9″.

17. Draw a trapezium *ABCD* with sides *AB* 2″, *BC* $3\frac{1}{2}$″, *CD* $2\frac{1}{2}$″, *DA* 2″, *AD*, *BC* being parallel. By measurement and calculation, find the area.

18. Repeat the trapezium in Question 17 and reduce it to an equal triangle.

19. The base *BC* of a triangle is trisected at *D* and *E*, and *AD* is joined (Fig. (1)). Why is $\triangle ADC$ half the area of $\triangle ABD$?

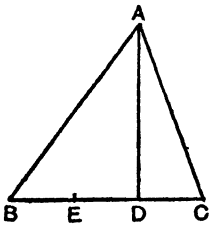


Fig. (1)

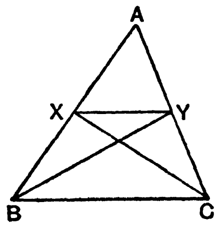


Fig. (2)

20. The sides *AB*, *AC* of a $\triangle$ *ABC* are bisected at *X*, *Y*. *BY*, *CX* are joined (Fig. (2)). Name two triangles equal in area to $\triangle$ *AXY*.

21. Since the triangles named in your answer to Question 20 are equal, what follows?

22. Draw a triangle with sides 4″, 3″, $3\frac{1}{2}$″. Make a rectangle equal to it.

23. Draw any five-sided figure and construct a quadrilateral equal to it.

24. Draw any quadrilateral. Reduce it to an equal triangle. Construct a rectangle equal to this triangle.

25. Draw a triangle with sides 2·8″, 2·5″, 2·3″. On a base 3·1″ construct a triangle equal in area to it.

26. Draw to a scale of 1 inch to 100 yards the field represented by the following extracts from the Field-Book, and calculate the area of the field to the nearest acre.

	Yards	
	D	
	800	
150	500	
	350	200
250	150	
	A	Go N.W.

27. A field *ABCDE* is surveyed with the following results:

	Yards to D	
	400	
	350	360 to C
420 to E	320	
	110	460 to B
	From A	

Draw the plan of the field to a scale of 1 inch to 200 yards, and determine the area of the field in acres and square yards.

28. Draw a plan of the field *ABCDE* from the Field-Book entry below, to a scale of 1 inch to 2 chains, and find the area of the field in acres.

	Links	
	To D	
	350	
250 to C	300	
	280	240 to E
210 to B	210	
	From A	

29. Draw a plan, and find the area in acres of a field with straight hedges from the following data:

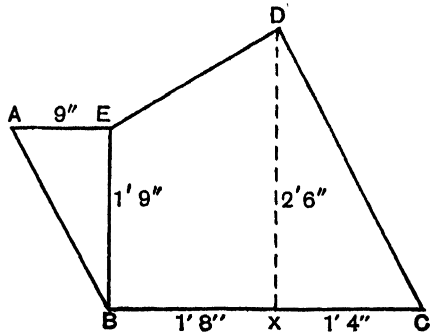
	Links	
	To C	
	350	
240 to D	250	
	150	120 to B
180 to E	100	
	From A	

30. Draw a plan, and find the area of a field with straight hedges from the following data:

	Links	
	To C	
20 to D	700	
	650	48 to B
92 to E	520	
70 to F	340	
	From A	

31. Find the area of the piece of sheet copper shown in the sketch. If 1 sq. ft. of the metal weighs 5·6 lb., find its weight in lb.

($\angle$s *AEB, EBC, DXC* are right angles.)



32. A quadrilateral $ABCD$ has the following dimensions: $AB = 240$ cm.; $BC = 150$ cm.; $AD = 110$ cm.; $\angle ABC = 85°$; $\angle BAD = 90°$. Draw a plan of the quadrilateral to a suitable scale and find its area to the nearest square metre.

33. A plot of land $ABCD$ has the following dimensions: $AB = 48$ ft.; $AD = 26$ ft.; $BC = 22$ ft.; $CD = 36$ ft.; and $\angle A = 75°$. Draw the plot to the scale of $\frac{1}{10}$ inch $= 1$ ft. Reduce the quadrilateral to a triangle having the same area, and find its area.

SIMILAR TRIANGLES*

43. In an earlier section it was discovered that two triangles need not necessarily be congruent when any three parts are equal in each. Thus two triangles with all angles equal, each to each, might be very different in size, though they would be similar in shape. Such triangles are known as similar triangles.

But although the corresponding sides of these triangles are not equal, they are definitely related to one another.

Let ABC, DBE be two such similar triangles with the smaller placed so as to occupy a corner of the larger (Fig. (1)). Suppose it is known that $BE = \frac{3}{5}$ of BC, or $\frac{BE}{BC} = \frac{3}{5}$.

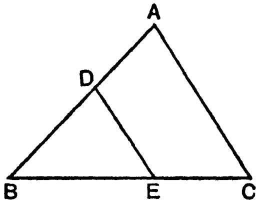


Fig. (1)

Divide BC into five equal parts as shown in Fig. (2) and draw parallels to AC through the points. DE will be one of these parallels, since $\angle DEB = \angle ACB$.

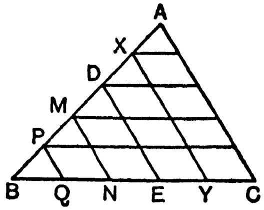


Fig. (2)

These four parallels will cut AB at four points. Through these four points draw parallels to BC.

We have now divided the figure ABC into a number of triangles and parallelograms, and all these parallelograms have one side equal to one of the divisions of BC.

* Paragraphs 43 to 48 and Exercise 23 may be omitted if desired.

The triangles are all equal because one side is either a side of one of these parallelograms or one of the divisions of BC.

$\therefore$ AB is divided into 5 equal parts, and $BD = 3$ of them.

$\therefore$ $BD = \frac{3}{5}$ of AB, or $\frac{BD}{AB} = \frac{3}{5}$.

Similarly AC is divided into 5 equal parts, and $DE = 3$ of them. $\therefore$ $DE = \frac{3}{5}$ of AC, or $\frac{DE}{AC} = \frac{3}{5}$.

Hence we find that the sides of these similar triangles are proportional to one another, i.e. corresponding sides are the same fractions of one another. This is always true, whatever the ratio between any pair of corresponding sides.

Learn:

The corresponding sides of similar triangles are proportional.

44. Areas of Similar Triangles.

Consider Fig. (2) again. It will be seen that each small parallelogram is twice the area of each small triangle, being of equal bases and equal heights.

DBE contains 3 parallelograms and 3 triangles; this is equivalent to 9 triangles.

ABC contains 10 parallelograms and 5 triangles; or 25 triangles.

$$\therefore \frac{DBE}{ABC} = \frac{9}{25} = \left(\frac{3}{5}\right)^2; \text{ and } \frac{BE}{BC} = \frac{3}{5}.$$

Consider MBN and XBY.

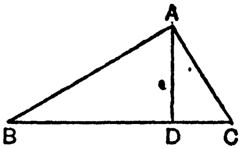
$$MBN = 1 \text{ parm.} + 2 \text{ triangles} = 4 \text{ triangles,}$$
$$XBY = 6 \text{ parm.} + 4 \text{ triangles} = 16 \text{ triangles.}$$
$$\therefore \frac{MBN}{XBY} = \frac{4}{16} = \left(\frac{2}{4}\right)^2, \text{ or } \left(\frac{1}{2}\right)^2; \text{ and } \frac{BN}{BY} = \frac{1}{2}.$$

These relations will be found to be true, whatever the ratio between two pairs of corresponding sides.

Learn:

The areas of similar triangles are proportional to the squares of corresponding sides.

45. An Important Exercise.



Let ABC be a triangle with a right angle at A. BC is called the *hypotenuse*.

Draw AD at right angles to BC.

The triangles ABD, ACD, ABC are all similar, with angles and sides corresponding as follows:

	Right angle	Smallest angle	Middle angle	Hypotenuse	Smallest side	Middle side
ABD	ADB	ABD	BAD	AB	AD	BD
ACD	ADC	DAC	ACD	AC	DC	AD
ABC	BAC	ABC	ACB	BC	AC	AB

$\angle BAD = \angle ACB$, because each is complementary to $\angle CAD$.

$\angle DAC = \angle ABC$, because each is complementary to $\angle BAD$.

$$\therefore \frac{ABD}{ABC} = \frac{AB^2}{BC^2} \text{ and } \frac{ACD}{ABC} = \frac{AC^2}{BC^2}.$$

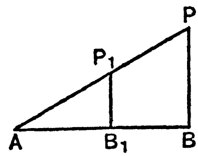
$$\therefore \frac{ABD + ACD}{ABC} = \frac{AB^2 + AC^2}{BC^2}.$$

But $\quad ABD + ACD = ABC. \qquad \therefore AB^2 + AC^2 = BC^2.$

Learn:

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

46. Relations between the sides of a right-angled triangle.



Let PAB be any angle.

Draw PB perpendicular to AB.

Then PBA is a right-angled triangle.

Let P_1B_1 be another such perpendicular. Then P_1AB_1 and PAB are similar triangles.

We have proved that the sides of similar triangles are proportional:

$$\therefore (1)\ \frac{P_1B_1}{AP_1} = \frac{PB}{AP}, \quad (2)\ \frac{AB_1}{AP_1} = \frac{AB}{AP}, \quad (3)\ \frac{P_1B_1}{AB_1} = \frac{PB}{AB}.$$

That is to say $\frac{PB}{AP}$, $\frac{AB}{AP}$ and $\frac{PB}{AB}$ always have the same value as long as the angle A remains the same.

Note that: AP is the hypotenuse,
PB is *opposite* the angle A,
AB is *adjacent* to the angle A.

The above fractions, or ratios, are known as the sine, cosine and tangent of the angle A.

I.e. The sine of the angle A (written sin A, but called sine A)

$$=\frac{\text{opposite side}}{\text{hypotenuse}}.$$

The cosine of the angle A (written and called cos A)

$$=\frac{\text{adjacent side}}{\text{hypotenuse}}.$$

The tangent of the angle A (written and called tan A)

$$=\frac{\text{opposite side}}{\text{adjacent side}}.$$

The mnemonic **sohcahtoa** will help you to remember these ratios, the initial letters giving the words sine, opposite, hypotenuse; cosine, adjacent, hypotenuse; tangent, opposite, adjacent.

47. Use of Sin, Cos, and Tan Tables.

The values of Sin, Cos, and Tan of any acute angle can readily be obtained from books of tables. The following are extracts from a page of tan tables:

	0′	10′	20′	30′	40′	50′	60′		Differences 1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	1·0000	1·0058	1·0117	1·0176	1·0235	1·0295	1·0355	44°	6	12	18	24	30	36	41	47	53
46	1·0355	·0416	·0477	·0538	·0599	·0661	·0724	43	6	12	18	25	31	37	43	49	55
47	1·0724	·0786	·0850	·0913	·0977	·1041	·1106	42	6	13	19	26	32	38	45	51	57
48	1·1106	·1171	·1237	·1303	**·1369**	·1436	·1504	41	7	13	20	27	**33**	40	46	53	60
49	1·1504	·1571	·1640	·1708	·1778	·1847	·1918	40°	7	14	21	28	34	41	48	55	62

From these tables we read tan 48° 45′ as follows:

(1) Look for 48° in the first column.

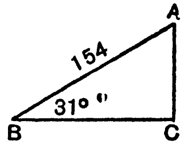
(2) Look along the horizontal line containing 48° until the vertical column for 40′ is met. Place the finger under ·1369. Note that this stands for 1·1369.

(3) Look further along this horizontal line until the vertical column for the remaining 5′ is met. The number found is 33.

(4) $1369 + 33 = 1402$.

$\therefore$ Tan 48° 45′ = 1·1402.

48. Use of the values of these ratios.



Ex. 1. $\angle ABC = 31°$, $\angle ACB = 90°$ and $AB = 154$ feet.

Find AC and BC.

$$(1)\quad \sin 31° = \frac{o}{h} = \frac{AC}{AB} = \cdot 5150.$$

$$\therefore \frac{AC}{154} = \cdot 5150,$$

$$AC = 154 \times \cdot 5150$$

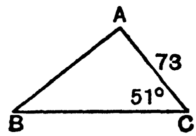
$$= 79 \cdot 3 \text{ ft.}$$

$$(2)\quad \cos 31° = \frac{a}{h} = \frac{BC}{AB} = \cdot 6009.$$

$$\therefore \frac{BC}{154} = \cdot 8572,$$

$$BC = 154 \times \cdot 8572$$

$$= 132 \text{ ft.}$$



Ex. 2. Here $\angle ACB = 51°$, $\angle BAC = 90°$ and $AC = 73$ ft.

Find AB.

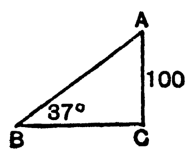
$$\text{Tan } 51° = \frac{o}{a} = \frac{AB}{AC} = 1 \cdot 2549.$$

$$\therefore \frac{AB}{73} = 1 \cdot 2549,$$

$$AB = 73 \times 1 \cdot 2549$$

$$= 91 \cdot 6 \text{ ft.}$$

Ex. 3. The angle of elevation of the top of a tower is 37° from a point on the ground. If the height of the tower is 100 feet, find the distance of the point from the foot of the tower.



Method 1. $\text{Tan } 37° = \frac{o}{a} = \frac{100}{BC} = \cdot 7536.$

$$\therefore BC = \frac{100}{\cdot 7536}$$

$$= 133 \text{ ft. to nearest foot.}$$

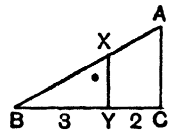
Method 2. It is clear that $\angle BAC = 53°$.

$$\text{Tan } 53° = \frac{o}{a} = \frac{BC}{100}.$$

$$\therefore \frac{100}{BC} = 1{\cdot}3270,$$

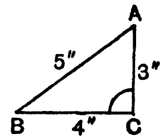
$$\therefore BC = 132{\cdot}7 \text{ ft.}$$

EXERCISE 23 (*a*). MENTAL.



1. XY is parallel to AC. $BY = 3''$, $YC = 2''$. If $AB = 6{\cdot}5''$, find BX.

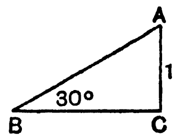
2. In the figure of Question 1, if $BY = 3$ cm., $YC = 2$ cm., $BX = \frac{3}{4}''$, find AX.



3. ABC is a right-angled triangle with sides $3''$, $4''$, $5''$, C being the right angle. Find the values of sin B, cos B, tan B.

4. In this figure find also cos A, sin A, tan A.

5. What can you deduce about the sin and cos of the complementary angles A and B?

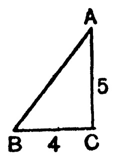


6. Sin 30° is known to be $\frac{1}{2}$ and $AC = 1''$. What is the length of AB?

7. Write down from tables the values of: sin 71°, sin 52°, sin 18°.

8. Write down cos 21°, cos 43°, cos 85°.

9. Write down tan 45°, tan 63°, tan 78°.



10. If $AC = 5$, $BC = 4$ and $\angle ACB = 90°$, find from tables the value of the angle A to the nearest degree.

EXERCISE 23 (*b*).

1. Draw any two triangles with corresponding angles equal. Find by measurement and calculation the ratios of the longest and shortest sides in each, and verify that they are equal.

2. Draw an angle of 56°. Measure the opposite side and the hypotenuse of a right-angled triangle containing this angle and hence find the sine of 56°. Compare this with the result obtained from tables.

3. Draw an angle of 73° and find its cosine in the same way.

4. Draw an angle of 15° and find its tangent.

5. A ladder is placed so that its foot is 12 feet from a wall. The angle which the ladder makes with the wall is 32°. Find the length of the ladder.

6. Show how to construct an angle whose cosine is $\frac{3}{5}$ and find by measurement its tangent.

7. Find by construction the sin, cos and tan of the angle 60°.

8. What is the cos of an angle whose sin is $\frac{5}{13}$?

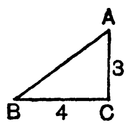
9. The string of a kite is 150 feet in length. If the string, supposed to be stretched tight, makes with the ground an angle of 53°, how high is the kite above the ground?

10. At a distance of 150 feet from the foot of a church tower, the elevation of the top of the tower was observed to be 43°. Find the height of the tower.

11. The sides of a rectangle are 8 and 10 inches. Find the angles made by a diagonal with a shorter side.

12. Find by tables the angles between the diagonals of a rectangle with sides 5 and 3 inches.

13. The sides of a right-angled triangle ABC containing the right angle are 3 and 4 units.



(1) Find $\angle B$ from tan tables.

(2) Find the cos of this angle.

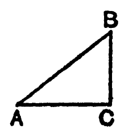
(3) Hence find AB.

(4) Find the value of $AC^2 + BC^2$ and compare it with AB^2.

14. Make any acute angle A. Find the sin and cos of the angle from tables, and hence obtain the value of $\sin^2 A + \cos^2 A$.

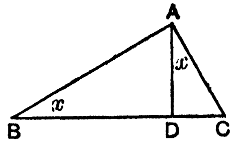
15. Make any acute angle A. Find the sin and cos by measurement and calculation and hence obtain the value of

$$\sin^2 A + \cos^2 A.$$



16. Obtain the result $\sin^2 A + \cos^2 A = 1$ by using the result $AC^2 + BC^2 = AB^2$ and dividing both sides by an appropriate quantity.

17. $\angle BAC$ is a right angle, and AD is perpendicular to BC.



Note that $\angle ABD = \angle CAD$.

(1) Find the values of $\sin x$ in $\triangle ADC$ and $\triangle ABC$ in terms of the sides. Hence prove $DC \cdot BC = AC^2$.

(2) Find the values of $\cos x$ in $\triangle ABD$ and $\triangle ABC$ in terms of the sides. Hence prove $BD \cdot BC = AB^2$.

(3) Add results of (1) and (2) and hence prove that $AC^2 + AB^2 = BC^2$.

Note.

This gives an additional proof of the fact that the square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

49. The following gives an alternative proof of the fact proved in paragraph 45 and in the last example of the previous exercise.

Make the squares on the sides BC, AB, AC.

Join KC, AD.

Draw AYX parallel to BD.

Steps in the proof.

(1) Square HB = twice $\triangle KBC$. Why?

(2) Rect. BX = twice $\triangle ABD$. Why?

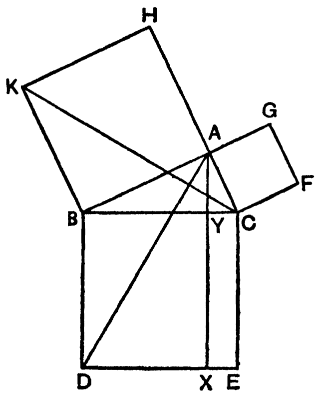
(3) $\triangle KBC = \triangle ABD$.—See below.

(4) $\therefore$ Sq. HB = rect. BX.

Similarly

(5) Sq. AF = rect. YE.

(6) $\therefore AB^2 + AC^2 = BC^2$.



Why is $\triangle KBC$ equal to $\triangle ABD$? Imagine the $\triangle ABD$ pivoted round about B until AB takes the position KB. Then BD will take the position BC because we have turned these sides through a right angle.

The following are familiar examples of right-angled triangles:

(1) With sides 3, 4, 5 units, since $5^2 = 3^2 + 4^2$.

(2) With sides 5, 12, 13 units, since $13^2 = 5^2 + 12^2$.

Other combinations giving right-angled triangles are 7, 24, 25; 9, 40, 41.

This is called the *Theorem of Pythagoras* from its discoverer. The converse is equally true. If, in a triangle, $c^2 = a^2 + b^2$, where a, b and c are the lengths of the sides, then the triangle is right-angled, and C is the right angle.

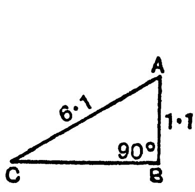
Ex. 1. The sides of a right-angled triangle which contain the right angle are 8·4 in. and 1·3 in. Find the length of the hypotenuse.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (8{\cdot}4)^2 + (1{\cdot}3)^2 \\ &= 70{\cdot}56 + 1{\cdot}69 \\ &= 72{\cdot}25, \\ \therefore\ c &= \sqrt{72{\cdot}25} = 8{\cdot}5 \text{ in.} \end{aligned}$$

Ex. 2. The hypotenuse of a right-angled triangle is 6·1 in., and one side is 1·1 in. Find the length of the other side.

$$\begin{aligned} c^2 &= a^2 + b^2, \\ \therefore\ a^2 &= c^2 - b^2 \\ &= (6{\cdot}1)^2 - (1{\cdot}1)^2 \\ &= 37{\cdot}21 - 1{\cdot}21 \\ &= 36, \\ \therefore\ a &= 6 \text{ in.} \end{aligned}$$

50. To obtain an approximate answer by construction we proceed as follows:



Draw a rough figure as shown.

To construct the correct figure first make AB 1·1″.

Draw BC at right angles to it.

With centre A and radius 6·1″ cut BC at C.

Then ABC is the required triangle, and $BC = 6$ in.

Ex. 3. Find graphically $\sqrt{7}$.

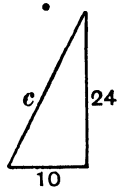
We require to find two squares of which the sum or difference is 7.

$$7 = 16 - 9.$$

If therefore we make a triangle with hypotenuse 4 and one side 3, the square of the other side will be 7,

$$\therefore \text{ the other side} = \sqrt{7}.$$

Ex. 4. A ladder with its foot 10 feet from the wall of a house just reaches a window 24 feet above the level of the ground. Find its length.



$$\begin{aligned} c^2 &= 24^2 + 10^2 \\ &= 576 + 100 \\ &= 676, \\ \therefore c &= \sqrt{676} = 26. \end{aligned}$$

The ladder is therefore 26 feet long.

The following example illustrates the arithmetical method of finding the square root of a number when the factor method is impossible.

Ex. 5. Find $\sqrt{357}$ to 2 places.

$$\begin{array}{r|l} & 3\,|57\cdot00|00|00(18\cdot894 \\ & 1 \\ \hline 28 & 257 \\ & 224 \\ \hline 368 & 3300 \\ & 2944 \\ \hline 3769 & 35600 \\ & 33921 \\ \hline 37784 & 167900 \end{array}$$

Sq. root $= 18\cdot89$ to 2 places.

Method.

(1) Mark off the figures in groups of 2, starting from the decimal point.

(2) Find the sq. root of the first group, viz. 3. This is 1.

(3) Square it and subtract result from 3. Bring down the next group. The result is 257.

(4) Double the number now in the sq. root, i.e. 1, and place it to the left of 257. By the side of this result, viz. 2, and by the side of the first figure in the quotient place such a number—here 8—that 28×8 will give the largest possible quantity to take from 257. Subtract.

(5) Repeat process (4), i.e. double 18 and place 36 to the left of the second remainder 3300. By the side of 36 and 18 place 8 because 8×368 gives the largest possible quantity to take from 3300.

(6) Proceed until the required number of places is obtained.

EXERCISE 24.

If a, b and c are the sides of right-angled triangles in which C is the right angle, find to the first place of decimals the missing side.

	a	b	c
1.	$2\frac{1}{2}''$	$6''$	
2.	$1\frac{1}{2}''$		$2\frac{1}{2}''$
3.		$1\frac{1}{4}''$	$3\frac{1}{4}''$
4.		$12''$	$12\frac{1}{2}''$
5.	3 cm.	$13\frac{1}{3}$ cm.	
6.	$8''$	$9''$	
7.		3·3 cm.	3·9 cm.
8.	$2{\cdot}3''$	$3{\cdot}5''$	
9.	231 mm.	309 mm.	
10.		5·37	$10{\cdot}74''$.

Construct graphically using $1''$ for unit:

11. $\sqrt{3}$. 12. $\sqrt{5}$. 13. $\sqrt{7}$. 14. $\sqrt{11}$.

15. The sides of a rectangle are 20 ft. and 25 ft. Find the length of the diagonal.

16. The side of a square is 13 cm. What is the length of the diagonal to the nearest mm.?

17. Find the side of a square which has a diagonal of 10 in., to the nearest tenth of an inch.

18. A ladder 29 ft. long just reaches a window 20 ft. above ground. How far out is the foot?

19. Find the height of an isosceles $\triangle$ which has a base $5''$ and two sides $6\frac{1}{2}''$.

20. What is the slant edge of a cone which has a vertical height of $17''$ and a base $9''$ in diameter?

21. Which of the following $\triangle$s are right-angled?

(*a*) Sides $2''$, $2\frac{1}{2}''$, $1\frac{1}{2}''$.

(*b*) Sides 5 cm., 12 cm., 14 cm.

(*c*) Sides $6''$, $1\frac{3}{4}''$, $6\frac{1}{4}''$.

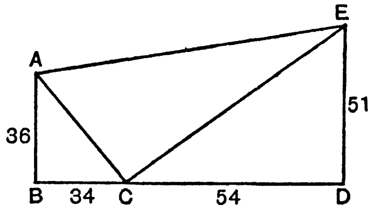
22. Construct a line whose length is $2\sqrt{3}$ cm.

23. The diagonals of a rhombus are $5''$ and $8''$ in length. Calculate the length of a side.

24. The sides of a quadrilateral $ABCD$ are AB 16 cm., BC 12 cm., CD 29 cm. and DA 21 cm. If $\angle ABC$ is a right angle, prove that $\angle CAD$ is also a right angle.

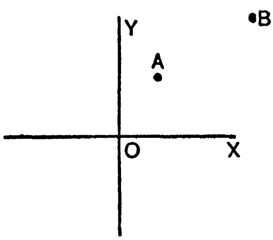
25. From the vertex A of a triangle ABC a perpendicular AX is drawn to the base. Prove that $AB^2 - AC^2 = BX^2 - XC^2$.

26. In the figure, which is not drawn to scale, AB and ED are perpendicular to BD. The sides being as shown, calculate AC^2, CE^2, AE^2 and then prove that $\angle ACE$ is a right angle.



27. A and B are points plotted on a graph. A is the point 2, 3 and B the point 7, 6.

Calculate the length of the line AB.



28. Similarly if A is the point -3, 4 and B the point 4, -3, calculate AB.

29. Is it possible to describe a quadrilateral with sides $8''$, $9''$, $1''$, $12''$ and with opposite angles right angles? If it is, calculate the length of one diagonal.

30. $ABCD$ is a quadrilateral in which the angles at B and C are right angles, AB is $1\frac{1}{2}''$, BC is $2''$, and CD is $3''$. Calculate the length of AD.

31. A ladder rests with one end on the ground and the other projecting over the top of a wall 36 ft. high. The end on the ground is 27 ft. away from the wall, and the ladder projects 15 ft. of its length beyond the top of the wall. Find by calculation the greatest distance from the wall at which one end may rest on the ground so that the other reaches to the top of the wall.

32. The squares on the sides of a triangle are of areas 1 sq. in., 1 sq. in. and 2 sq. in. respectively. *Calculate* (without drawing an accurate figure) (i) the angles, (ii) the area of the triangle.

33. A rectangular field with an area of 30,618 sq. yd. has a width of 94·5 yd. What is the length of a straight path crossing it from corner to corner?

34. A ladder 52 ft. long was placed so as to reach a window 48 ft. high. On turning the top of the ladder over to the other side of the street, keeping its foot in the same place, it reaches a window 20 ft. high. Find the width of the street.

35. The perimeter of a square field is a quarter of a mile. Draw a plan of the field to a scale of 1 inch to 1 chain. Calculate the length of the diagonal.

SECTIONAL REVISION C

EXERCISE 25 (*a*). MENTAL.

1. What is the formula for the area of a parallelogram?

2. The hypotenuse of a rt.-angled triangle is $2\frac{1}{2}''$ long and one side is $1\frac{1}{2}''$. Find the other side.

3. Calculate the area of the quadrilateral from the dimensions given (Fig. (1)).

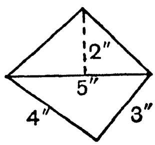


Fig. (1)

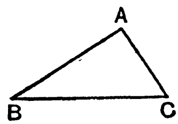
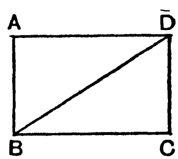


Fig. (2)

4. How would you bisect the triangle *ABC* by a line drawn through the vertex? (Fig. (2).)

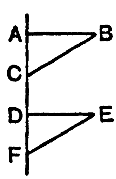
5. What facts do you know about the diagonals of a rhombus?

6. Why is $\triangle$ *DBC* half the area of the rectangle *ABCD*?



7. If half the vertical angle of an isosceles triangle $= 25°$, what is each base angle?

8. A set-square *ABC* is slid down a ruler to the position *DEF*. Why is *BC* parallel to *EF*?



EXERCISE 25 (*b*).

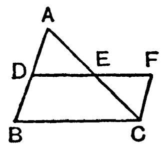
1. Construct a triangle with sides 3″, 3·5″, 4·5″, and find its area.

2. Calculate the area of a trapezium with parallel sides 8·52 cm. and 5·48 cm. which are 5 cm. apart.

3. Make a parallelogram with sides 4″, 3″, and an angle of 78°. Then construct a rectangle equal to it in area.

4. Copy the parallelogram in Question 3 and reduce it to an equal triangle.

5. Bisect the two sides AB, AC of a $\triangle$ at D and E.
Produce DE to F making $EF = DE$. Join FC.
Prove $\triangle$'s ADE, CFE congruent.



6. From the results in Question 5, how do you know that $DBCF$ is a parallelogram?

7. What is the slant edge of a cone if the vertical angle is 60° and the diameter of the base is 4″?

8. State the three main sets of conditions for the congruence of two triangles.

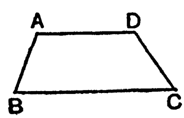
EXERCISE 26 (*a*). Mental.

1. Give the formula for the area of a triangle with base b and vertical height h. In how many ways can the area of a triangle thus be calculated?

2. The sides of a triangle are in the ratio $\frac{3}{4} : 1 : 1\frac{1}{4}$. How do you know it is right-angled?

3. What is the area of a trapezium in which the parallel sides are 1·56″ and 2·44″ and the distance between them 4″?

4. How would you divide the trapezium $ABCD$ into a parallelogram and a triangle?



5. What facts do you know about the sides and angles of a parallelogram?

6. If $\angle BAD = 118°$ and $\angle ADC = 108°$, and $\angle$s at B and C are bisected by BX and CX, calculate $\angle BXC$. (Fig. (1).)

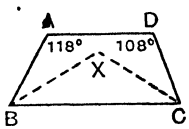


Fig. (1)

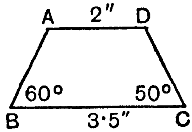
Fig. (2)

7. If these $\triangle$s are accurately constructed, are they congruent and why? (Fig. (2).)

8. What is the angle between the hands of a clock at 6.30?

EXERCISE 26 (*b*).

1. Find by measurement and calculation the area of the trapezium shown in the rough diagram.



2. Calculate the area of a parallelogram with sides 5″, 4″, and one angle 150°.

3. Make any parallelogram, and then construct another equal to it in area with a side twice as great as the longer side of the first.

4. Copy the trapezium in Question 1 and reduce it to an equal triangle.

5. If $\angle ABC = \angle DCB$ and $\angle BAD = \angle ADC$ (Fig. (3)), prove $AB = DC$.

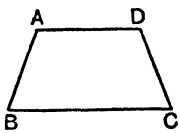


Fig. (3)

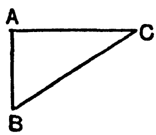


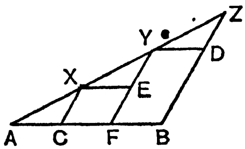
Fig. (4)

6. ABC is a rt.-angled triangle (Fig. (4)). $AB = \frac{1}{2}BC$ and $AC = 4''$. Calculate AB to 2 places of decimals.

7. Prove that the bisectors of the angles of a triangle meet at a point.

8. Prove the following method for the trisection of AB.

Draw any line AZ and along it cut off equal portions $AX.XY.YZ$. Draw XC, YF parallel to ZB.



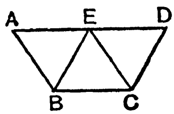
EXERCISE 27 (*a*). MENTAL.

1. Give the formula for the area of a trapezium if x and y are the lengths of the parallel sides and d the distance between them.

2. If the side of an equilateial triangle is 4 inches long, find the area of the square on the altitude.

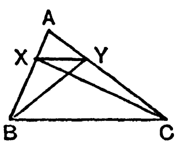
3. If $\sqrt{3} = 1{\cdot}73$ to 2 places, calculate the area of the triangle in Question 2 correct to 1 place of decimals.

4. If $ABCE$, $EBCD$ are parallelograms and AED is a straight line, how do you know at once that $ABCE = EBCD$?



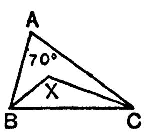
5. What fact do you know about the exterior angle of any triangle?

6. XY is drawn parallel to BC.



How do you know that $\triangle\, AYB = \triangle\, AXC$?

7. $\angle\, BAC = 70°$ and $\angle$s at B and C are bisected by BX, CX. Calculate $\angle\, BXC$.



8. What conditions must hold between the parts of two right-angled triangles if they are to be congruent?

EXERCISE 27 (*b*).

1. Find the area of the parallelogram which has a base 8·5 cm., one side 4·7 cm. and base angles 115° and 65°.

2. Calculate the area of a triangle with sides 5·2″, 5·7″, 6·1″.

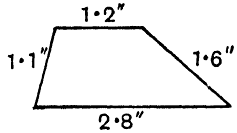
3. Make the triangle indicated in Question 2 and then construct a triangle equal to it on a base twice as great.

4. Make a triangle with a base 4″ and base angles 65°, 78° and construct a rectangle equal to it in area.

5. A right-angled triangle has an angle of 45°, and one side is 4″ long. Calculate the length of the hypotenuse to two places of decimals.

6. Construct the trapezium indicated in the diagram and measure its altitude.

7. Reduce the figure to an equal triangle and then make a rectangle equal to it.

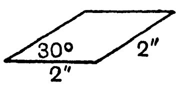


8. What is the size of each angle of a regular polygon of 120 sides?

EXERCISE 28 (*a*). MENTAL.

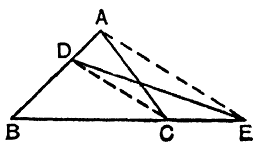
1. Give the formula for finding the area of a triangle given the length of the three sides, and say what each letter stands for.

2. What is the area of this rhombus?



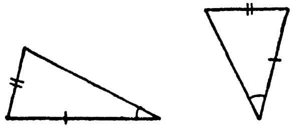
3. The hypotenuse of a right-angled triangle is 13 cm., and one side is 12 cm. What is the length of the other side?

4. If AE is parallel to DC, why is the triangle ABC equal in area to the triangle DBE?



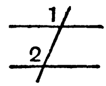
5. What fact do you know about the diagonals of any parallelogram?

6. If these $\triangle$s are constructed accurately, are they congruent and why?

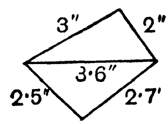


7. What is the vertical angle of a triangle if half the sum of the base angles is 55°?

8. What are $\angle$s 1 and 2 called with respect to one another?



EXERCISE 28 (*b*).



1. Find by measurement and calculation the area of the quadrilateral shown in the rough diagram.

2. Calculate the area of a rhombus which has a side of 5″ and one diagonal 6″.

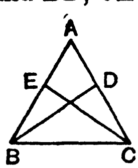
3. Make a triangle with sides 2·5″, 2·9″, 3·7″, the last being the base, and then construct a triangle equal to it in area on a base 4·5″.

4. Construct any five-sided rectilineal figure and reduce it step by step to an equal triangle.

5. An isosceles triangle has two sides 5·5 cm. long and a base 4·8 cm. Calculate the altitude and hence the area.

6. Construct a rectangle with sides 3·5″ and 2·8″. Make a rectangle equal to it on a base half the length.

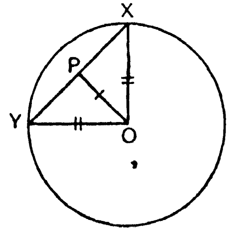
7. *ABC* is equilateral and *BD*, *CE* two altitudes. Prove that $BD = CE$.



8. The angles of a quadrilateral are $2x - 5°$, $3x + 25°$, $4x + 75°$, and 175°. Find x.

THE CIRCLE

51. The essential feature of a circle is that every point on the circumference is the same distance from the centre. This distance is the radius.



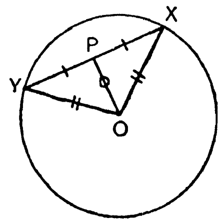
The perpendicular to a chord from the centre of a circle bisects the chord.

Let XY be the chord, O the centre, OP the perpendicular.

We want to prove $PX = PY$.

At a glance we see that the hypotenuse and one side are equal in the two triangles.

$$\therefore PX = PY.$$



Conversely if a line OP from the centre bisects XY then OP is at right angles to XY.

In this case the Δs OPX, OPY have 3 sides equal in each.

$$\therefore \angle OPX = \angle OPY.$$

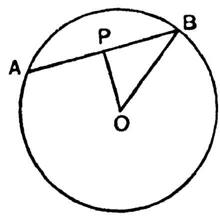
But together they make two right angles,

$\therefore$ each is a right angle.

Learn:

A straight line drawn from the centre of a circle to bisect a chord which is not a diameter is at right angles to the chord.

The perpendicular to a chord from the centre bisects the chord.



52. Ex. 1. A chord 3″ long is drawn in a circle of radius $2\frac{1}{2}''$. Find the distance from the centre.

OP bisects AB.

$$\therefore PB = 1\tfrac{1}{2}'',$$
$$OB = 2\tfrac{1}{2}''.$$
$$OP^2 = OB^2 - PB^2$$
$$= (2\tfrac{1}{2})^2 - (1\tfrac{1}{2})^2$$
$$= \frac{25}{4} - \frac{9}{4} = \frac{16}{4} = 4.$$
$$\therefore OP = 2''.$$

Ex. 2. A chord of a circle of radius $6\frac{1}{2}''$ is $6''$ from the centre. Find its length.

Using the former figure,

$$OB = 6\tfrac{1}{2}'', \quad OP = 6''.$$

$$\therefore PB^2 = OB^2 - OP^2$$

$$= (6\tfrac{1}{2})^2 - (6)^2 = \frac{169}{4} - 36 = \frac{169 - 144}{4} = \frac{25}{4}.$$

$$\therefore PB = \frac{5}{2}.$$

Hence the chord $= 5''$.

Ex. 3. A chord $6''$ long is $4''$ from the centre. Find the radius.

$$\text{Here } PB = 3,\ OP = 4.$$

$$\therefore OB^2 = 9 + 16 = 25,$$

$$OB = 5.$$

From Ex. 1 we see that wherever the chord of length $3''$ is drawn in the circle the distance from the centre is the same.

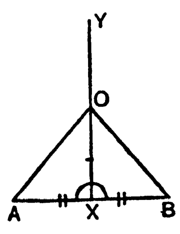
From Ex. 2 we see that all chords $6''$ from the centre must be of the same length.

Learn:

Equal chords in a circle are equally distant from the centre.

Chords equally distant from the centre of a circle are equal.

53. The centres of all circles passing through two points lie on the perpendicular bisector of the line joining the two points. This is proved as follows:



Let AB be the line joining the two points and XY the perpendicular bisector.

Take any point whatever on XY and join OA, OB.

The triangles OXA, OXB are congruent, having 2 sides and the contained angle equal in each.

$$\therefore OA = OB.$$

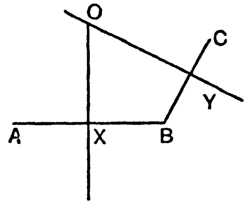
This is true for any point O.

$\therefore$ with centre O and radius OA a circle can be described passing through A and B.

Learn:

The centres of all circles passing through any two points lie on the perpendicular bisector of the line joining the two points.

54. Ex. 1. Describe a circle passing through three given points A, B, C.



The circle must pass through A and B.

$\therefore$ the centre lies on the perp.-bisector OX.

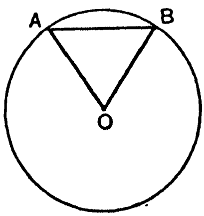
Also the circle passes through B and C.

$\therefore$ the centre lies on OY.

Hence the centre must be at O, and there is only one such centre.

This gives us a method of constructing a circle passing through the vertices of a triangle.

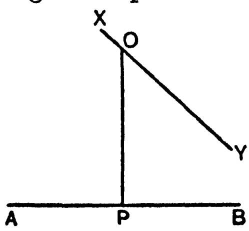
Ex. 2. Construct a circle of radius 1″ passing through two points $\frac{1}{2}$″ apart.



From the rough figure we see that in order to find the centre of the circle we must make the triangle ABO with AB $\frac{1}{2}$″, AO 1″, BO 1″.

Then with centre O and radius 1″ the circle can be described passing through A and B.

Ex. 3. Construct a circle passing through two points and having its centre on a given line.



Let A, B be the points, XY the line.

The centre must lie on the perpendicular bisector of AB.

i.e. it lies on OP.

This line cuts XY at O.

$\therefore$ O is the centre of the circle.

EXERCISE 29.

1. A chord 5″ long is 6″ from the centre. Find the radius of the circle.

2. How far from the centre of a circle of radius 17 cm. is a chord 15 cm. in length?

3. What is the length of a chord in a circle if the radius is 18 cm. and the distance of the chord from the centre 8 cm.?

4. XY is a chord 5″ long in a circle of radius $6\frac{1}{2}$″. If O is the centre of the circle, what is the area of the triangle OXY?

5. Two points A and B are 5 cm. apart. Draw a circle of radius 3 cm. passing through A and B.

6. A chord of a circle is 2·3″ long and the distance from the centre is 1·2″. Construct the circle and measure and calculate the radius.

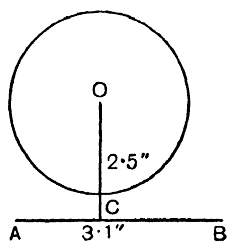
7. Three points A, B, C are such that $AB = 2{\cdot}3''$, $BC = 1{\cdot}8''$, and $CA = 1{\cdot}7''$. Describe a circle passing through the three points.

8. Describe a triangle with sides 3, 4 and 5 inches and construct the circumscribing circle.

9. Draw a straight line AB 3·1″ and fix a point O so that OC is perpendicular to AB and of length 2 5″, and AC 1·2″.

With centre O and radius 1·8″ describe a circle.

Construct a circle passing through A and B with its centre on the given circle.



10. Draw a circle of radius 1·8″. Fix a point A 1·3″ from the centre and through O draw a chord 3″ long.

11. The length intercepted on a certain straight line by a circle of radius 3″ is 5″. What will be the length intercepted on the same straight line by a concentric circle of radius 5″?

12. O is the mid-point of the line joining the centres of two equal intersecting circles, and P is one of the points in which they intersect. If the radius of each is 3″ and the line joining the centres is 5″, calculate the length of OP.

13. O is the centre of a circle circumscribing the triangle ABC; prove that if AO bisects BC, the angle ABO equals the angle ACO.

14. Prove that the curve joining all the mid-points of equal chords of a circle is a concentric circle, and say how you would discover its radius.

·55. Angles in a Circle.

AOB is an angle at the centre of a circle standing on the arc AB.

ACB is an angle at the circumference standing on the same arc.

It can be proved that $\angle AOB =$ twice $\angle ACB$.

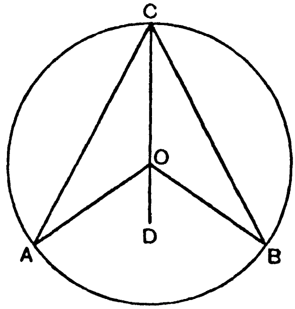


Fig. (1)

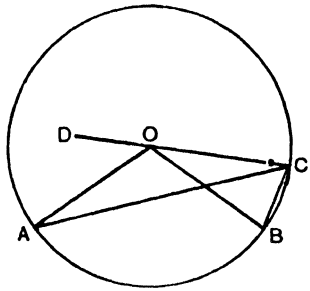


Fig. (2)

Proof. OAC is an isosceles triangle.

$$\therefore \angle OAC = \angle OCA.$$

Hence any angle equal to the sum of these two angles is twice either of them.

$\angle AOD$ is such an angle, for the exterior angle of any triangle equals the sum of the two interior opposite angles.

$$\therefore \angle AOD = 2\angle ACO. \qquad \ldots\ldots(a)$$

In the same way $\angle BOD = 2\angle BCO.$ ......(b)

In Fig. (1), by adding (a) and (b) we get $\angle AOB = 2\angle ACB$.

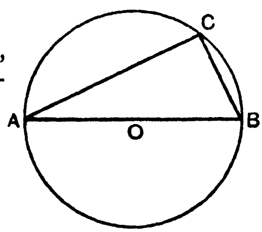
In Fig. (2) we subtract and get the same results.

Learn:

The angle at the centre of a circle is twice the angle at the circumference standing on the same arc.

56. Particular Cases and Extensions.

(1) *Suppose $\angle AOB$ is a straight angle,* i.e. AOB is a diameter and ACB is a semi-circle.



It is still true that $\angle AOB = 2\angle ACB$.

But $\angle AOB = 2$ rt. $\angle$s.

$\therefore \angle ACB$ is a right angle.

Learn:

The angle in a semi-circle is a right angle.

(2) In Fig. (1) $\angle ACB$ *is obtuse* and the reflex $\angle AOB$, i.e. an angle more than 2 rt. $\angle$ s, is twice $\angle ACB$.

The same proof as before will hold good.

$$\text{For} \quad \angle AOD = 2\angle ACD.$$
$$\text{And} \quad \angle BOD = 2\angle BCD.$$
$$\therefore \text{ Reflex } \angle AOB = 2\angle ACB.$$

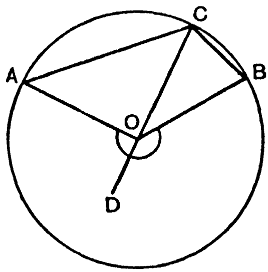


Fig. (1)

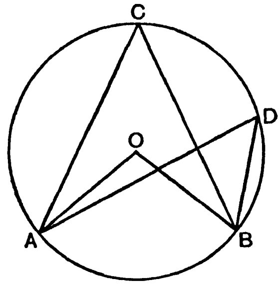
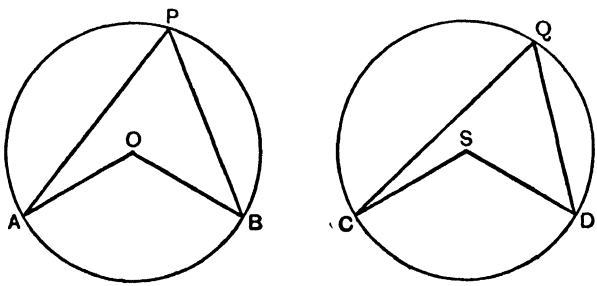


Fig. (2)

(3) In Fig. (2) *we have two* $\angle$ s ACB, ADB *on the same arc* AB, i.e. *in the same segment* $ACDB$.

$\angle AOB$ = twice either angle. $\quad\therefore \angle ACB = \angle ADB$.

(4) *The two circles in the figures given below are of equal radii and* arc AB = arc CD.

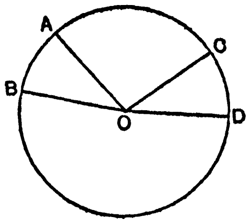


Hence $\angle APB$ and $\angle CQD$ stand on equal arcs, or are in equal segments.

It is possible to place one circle on the other so that arcs CD, AB coincide, and the radii also coincide.

$\angle APB$ and $\angle CQD$ will be in the same segment.

$$\therefore \angle APB = \angle CQD.$$

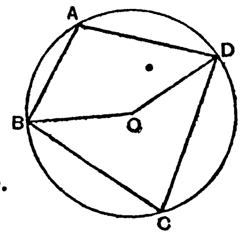


(5) Let AB and CD be equal arcs in the same circle.

We can imagine OCD to be revolved about the centre O so that the equal arcs CD, AB coincide.

$$\therefore \angle AOB = \angle COD.$$

Similarly the angles at the circumference standing on these equal arcs will be equal.



(6) $ABCD$ is a quadrilateral in a circle.

$$\angle BOD = 2 \angle BCD.$$

Reflex $\angle BOD = 2 \angle BAD.$

Adding, we get:

$\angle BOD +$ Reflex $\angle BOD$

$$= 2 (\angle BCD + \angle BAD).$$

$\therefore$ 4 rt. $\angle$s $= 2$ (opp. $\angle$s of the quadl.).

$\therefore$ Opposite angles of a quadrilateral inscribed in a circle (or cyclic quadrilateral) are equal to two right angles.

Learn:

1. *Angles in the same segment of a circle are equal.*

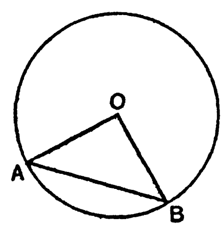
Or *Angles at the circumference of a circle which stand on the same arc are equal.*

2. *In equal circles angles, whether at the centre or the circumference, which stand on equal arcs are equal.*

3. *Angles at the centre or circumference of a circle which stand on equal arcs are equal.*

4. *The opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.*

EXERCISE 30.



1. If $\angle AOB$ at the centre is 102°, calculate

(1) $\angle OAB$,

(2) Reflex $\angle AOB$.

2. Using the figure of Question 1, what is the size of the angle APB at the circumference standing on the arc AB?

3. If the point P is in the minor segment of this circle, what is the angle APB?

4. If $\angle APB$ in Fig. (1) is 72°, find

(1) Reflex $\angle AOB$, O being the centre.

(2) $\angle ARB$.

(3) $\angle OAB$.

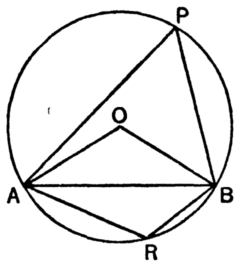


Fig. (1)

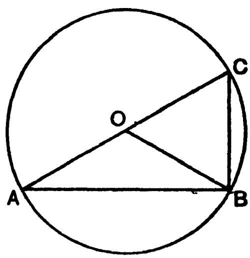


Fig. (2)

5. If $\angle AOB$ in Fig. (2) is 118°, and AOC is a straight line, find

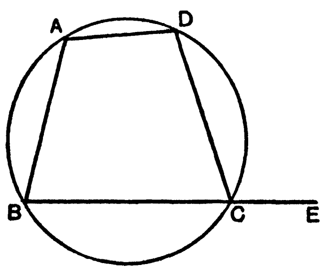
(1) $\angle OAB$.

(2) $\angle OBC$.

6. Using the figure of Question 5, if $\angle OBA$ is 27° find $\angle ACB$.

7. Using the same figure, if $\angle OBC$ is 61° find $\angle AOB$.

8. If $\angle BAD$ in this quadrilateral is 105°, find $\angle DCE$.



9. Two adjacent angles of a cyclic quadrilateral are 71° and 89°. What are the other two?

10. If the median (i.e. the line from the vertex bisecting the base) of a triangle is half the base, show that the triangle must be right-angled.

11. AB and CD are parallel chords in a circle. Show that $AD = BC$.

12. AB is a diameter of a circle and CD a chord at right angles to it cutting it at E. If another chord AFG is drawn cutting CD at F, prove that a circle can be drawn to pass through $B, E, F,$ and G.

13. Show how to construct a triangle ABC, given the length of BC, the angle at A, and the area of the triangle. (Use the fact that the angle BOC at the centre of the circumscribing circle is twice $\angle A$.)

14. Construct a quadrilateral $ABCD$, which can be inscribed in a circle, and is such that AB is 2″, the angle ABC is a right angle, the angle BAD is 70°, and the side DC produced meets AB beyond B at a point distant 1″ from B.

15. Construct a right-angled triangle with a hypotenuse 4 inches long and a side 2 inches long. Explain your construction and calculate the area of the triangle.

16. A hexagon is inscribed in a circle; prove that the sum of three alternate angles of the hexagon is equal to four right angles.

17. $ABCDE$ is a pentagon inscribed in a circle and all its angles are equal; prove that all its sides are equal.

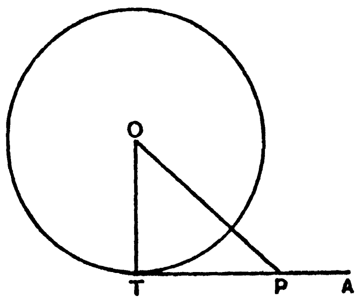
57. Tangents.

A tangent to a circle touches it at one point only.

Let AT be such a tangent.

OT is obviously the shortest distance from O to AT, for any other line OP must be greater than a radius.

But the shortest distance from a point to a line is the perpendicular distance.



$\therefore \angle OTA$ is a rt. $\angle$.

Hence to draw a tangent at T we simply make $\angle OTA$ a rt. $\angle$.

Learn:

The tangent at any point of a circle and the radius through the point are perpendicular to one another.

58. To draw a tangent to a circle from a point outside it.

Draw a rough figure showing the tangent AT and the radius OT. We know

(1) That $\angle ATO$ must be a rt. $\angle$.

(2) That the angle in a semi-circle is a right angle.

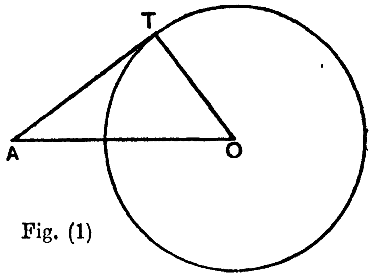


Fig. (1)

Hence we make a semi-circle on AO, and where this cuts the circle at T we get the point of contact of a tangent AT. (Fig. (1).)

In the diagram Fig. (2), it is seen that the semi-circle may be constructed on either side of the diameter.

Hence both S and T can be points of contact of tangents from A.

Note too that these tangents are equal, because Δ s AOT, AOS are congruent. Why?

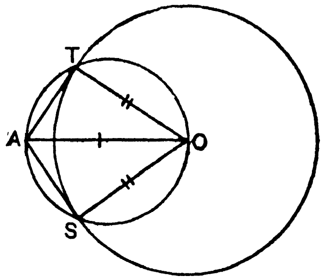


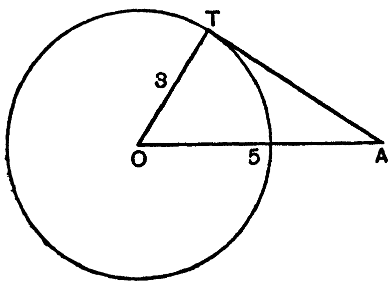
Fig. (2)

Learn:

(1) *Two tangents can be drawn from a point outside a circle.*

(2) *The tangents to a circle from an external point are equal.*

Ex. 1. Find the length of the tangent to a circle of radius 3″ from a point 5″ from the centre.



Draw a rough figure.

$\angle OTA$ is a rt. $\angle$.

$\therefore AT^2 = OA^2 - OT^2$

$= 25 - 9$

$= 16.$

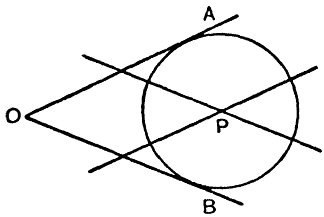
$\therefore AT = 4$ inches.

Ex. 2. If a tangent to a circle from an external point is 7″ long and the radius is 3″, how far is the point from the centre?

Here OA is required.

$$OA^2 = OT^2 + TA^2$$
$$= 3^2 + 7^2 = 9 + 49 = 58.$$
$$\therefore OA = \sqrt{58} = 7{\cdot}6''.$$

Ex. 3. Describe a circle of 1″ radius touching two given lines.



Draw lines parallel to OA and OB 1″ away. Then P is the centre.

59. Common Tangents to Two Circles.

Direct common tangents.

We see from the rough figure that at least two common tangents can be drawn to two circles, as long as one does not lie inside the other.

Those in the figure are called Direct Common Tangents.

Consider the tangent AB.

We have AX and BY both at rt. $\angle$ s to AB.

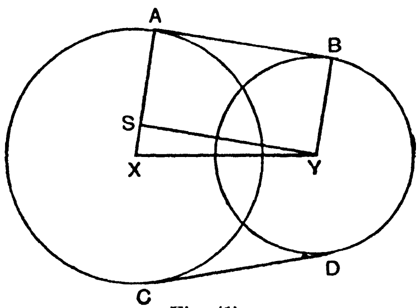


Fig. (1)

By drawing SY at rt. $\angle$ s to AS, we form a rectangle and a right-angled $\triangle$ XSY.

In the latter

SY = the common tangent AB,

XY is the line joining the centres of the circles,

$SX = AX - AS$

$= AX - BY$

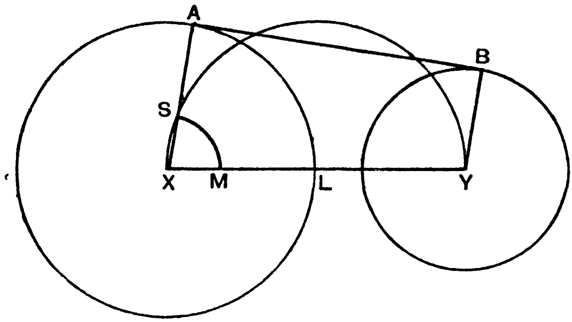
= the difference of the radii.

This gives us the method of construction.

Join XY, and on it describe a semi-circle.

The point S in the former figure must lie on this semi-circle.

Cut off XS equal to the difference of the radii.



Note that this can be done as follows:

(1) Make LM equal to the smaller radius.

(2) With centre X and radius XM cut the semi-circle at S.

Join XS and produce till it meets the circle at A.

This is one point on the common tangent.

Draw YB parallel to XA.

This gives the other point B.

Join AB.

No proof is required, because we have seen that the construction must inevitably produce the desired result.

Transverse common tangents.

From the rough figure we see that circles which do not touch or cut can have two transverse as well as two direct common tangents.

Consider the tangent AB.

It is at right angles to AX and BY.

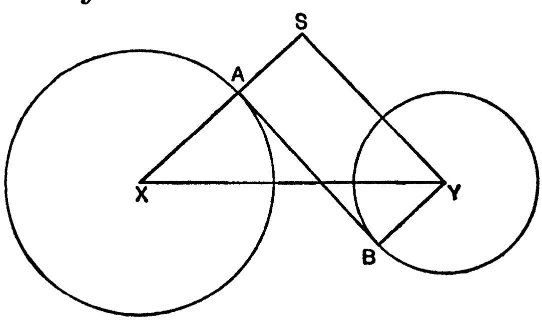


Fig. (2)

By drawing SY at right angles to XA produced, we form a rectangle and a right-angled $\Delta\ XSY$.

In the latter

$$\begin{aligned} SY &= \text{the common tangent } AB, \\ XY &\text{ is the line of centres,} \\ SX &= AX + AS \\ &= AX + BY \\ &= \text{the sum of the radii.} \end{aligned}$$

This gives us the method of construction, which is left as an exercise to the student.

Lengths of Common Tangents.

1. *Direct common tangent.*

In Fig. (1)

$$\begin{aligned} AB &= SY, \\ SY^2 &= XY^2 - SX^2 \text{ in the rt. } \angle\text{d. } \Delta\ XSY, \\ SY &= \sqrt{XY^2 - SX^2}, \\ t_1 &= \sqrt{c^2 - (a-b)^2}, \end{aligned} \qquad \ldots\ldots(a)$$

where c is the line of centres and a, b the radii.

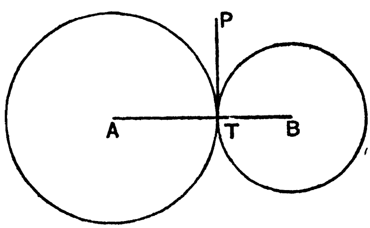
2. *Transverse common tangent.*

In Fig. (2)

$$\begin{aligned} AB &= SY, \\ SY^2 &= XY^2 - SX^2, \\ SY &= \sqrt{XY^2 - SX^2}, \\ t_2 &= \sqrt{c^2 - (a+b)^2}. \end{aligned} \qquad \ldots\ldots(b)$$

Results (a) and (b) should be learnt by heart.

60. Tangential Circles.



Let the two circles touch at T. There can only be one transverse common tangent.

Let it be TP.

TP is at right angles to TB and to TA.

$\therefore \angle ATP + \angle BTP = 2$ rt. $\angle$s.

$\therefore ATB$ is a straight line.

Learn:

If two circles touch, the point of contact lies on the straight line through the centres.

Ex. 1. Two circles of radii 8″ and 3″ have centres 10″ apart. Find the lengths of the common tangents.

(1) *Direct.*

$$\begin{aligned} t &= \sqrt{c^2 - (a-b)^2} \\ &= \sqrt{100 - 25} \\ &= \sqrt{75} = 5\sqrt{3} = 8{\cdot}7''. \end{aligned}$$

(2) *Transverse.*

$$\begin{aligned} t &= \sqrt{c^2 - (a+b)^2} \\ &= \sqrt{100 - 121} \\ &= \sqrt{-21}. \end{aligned}$$

This is not a real quantity and so we cannot draw transverse common tangents. An approximate figure will show why this is so.

Ex. 2. Two circles of radii 6″ and 2″ have centres 3″ apart. Can any tangents be drawn?

(1) Try
$$\begin{aligned} t &= \sqrt{c^2 - (a-b)^2} \\ &= \sqrt{9 - 16} \\ &= \sqrt{-7}. \end{aligned}$$

∴ No direct tangents can be drawn.

(2) Try
$$\begin{aligned} t &= \sqrt{c^2 - (a+b)^2} \\ &= \sqrt{9 - 64} \\ &= \sqrt{-55}. \end{aligned}$$

∴ No transverse tangents can be drawn. Of course not, because one circle lies wholly inside the other.

Ex. 3. How far apart are the centres of two circles of radii 5″ and 4″ if the direct common tangent is 10″ long?

$$\begin{aligned} t &= \sqrt{c^2 - (a-b)^2}, \\ 10 &= \sqrt{c^2 - 1}, \\ 100 &= c^2 - 1, \\ c^2 &= 101, \\ c &= \sqrt{101}. \end{aligned}$$

EXERCISE 31.

1. Draw a circle of radius 1·5″. Find a point P 3·1″ from the centre. From P draw a tangent to the circle.

2. Measure and calculate the length of this tangent.

3. What is the radius of a circle if the length of a tangent from a point 2″ from the centre is equal to 1·2″?

4. Find the distance from the centre of a circle of the point from which a tangent 2·5″ long can be drawn, if the radius of the circle is ·8″ long. Verify by construction.

5. Two concentric circles are drawn of radii 1·5″ and 1·9″. Find the length of a chord of the larger circle which touches the inner. Verify by construction.

6. The vertical angle of an isosceles triangle is 60°, and a circle of radius 3″ is inscribed in it. Find the distance of the centre of the circle from the vertex of the triangle.

7. Two lines AB, AC, each 2·8″ long, are drawn enclosing an angle of 78°. Describe a circle which touches these lines at B and C.

8. If AP and AQ are tangents to a circle with centre O, prove

$$\angle OAP = \angle OAQ.$$

9. O is the centre of a circle with radius 2·3″. P is an external point 1·2″ radially from the circumference. Calculate to the nearest hundredth of an inch the length of the tangents from P to the circle.

10. Two circles touch at A and from A a common tangent is drawn to the two circles. If any point P is taken on this line and tangents are drawn to the two circles, prove that these tangents are equal.

11. Construct two direct common tangents to two circles of radii 2″ and 1″ with centres 2·5″ apart.

12. Calculate the lengths of these tangents. Could transverse common tangents be drawn to these circles?

13. Construct two transverse common tangents to two circles of radii 1·5″ and 1·2″ with centres 3·5″ apart.

14. Calculate the lengths of these tangents.

15. What is the length of the direct common tangents to circles of radii 1·5″ and 3·5″ if the *circles* are 2″ apart radially?

16. If the length of the transverse common tangent to two circles of radii 3″ and 4″ is 1·2″, calculate how far the centres are apart.

17. Two engine wheels on the same side of an engine have their axles 10 feet apart, and their radii are 2′ 9″ and 2′ 2″. What length of railway line do they cover?

18. A see-saw is constructed from a 10 ft. plank and a portion of a tree-trunk which has a radius of 1′ 2″. If the plank is fastened at its middle point to the trunk, what is the distance from the end of the plank to the centre of the trunk?

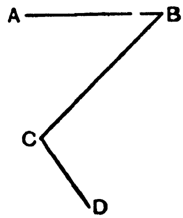
19. A sphere of radius 3 cm. is fitted into a cone whose semi-vertical angle is 30°. Find the distance of the centre of the sphere from the vertex of the cone.

20. Construct a circle of radius $1\frac{1}{2}''$ to touch two circles whose radii are $2''$ and $2\frac{1}{2}''$ and whose centres are $3\frac{1}{2}''$ apart.

21. Draw a line AB 4″ long and at A erect a perpendicular AC 3″ long. Describe a circle to pass through A, B and C. Produce AB to D, making BD 2″. From D draw tangents to the circle and measure them.

22. Construct a circle with radius 3·75 cm. Draw a diameter AOB where O is the centre of the circle. Bisect AO at C. Through C draw a line DC at right angles to AB and meeting the circle at D and E. Measure the length of the chord DE. At D and E draw tangents to the circle.

23. Draw three lines AB, BC, CD as shown in the figure, so that $BC = 2''$, $\angle ABC = 70°$, $\angle BCD = 100°$. Connect AB and BC by a tangential arc of $\frac{1}{2}''$ radius, and BC, CD by a tangential arc of $\frac{3}{4}''$ radius; and line in clearly the resulting figure.



61. The Circumference of a Circle.

It can be discovered by measurement that the ratio of the circumference of a circle to its diameter is always the same, however large the circle may be. This ratio is known as π (pi). Its value is approximately $3\frac{1}{7}$, or, more closely 3·14159.

$$\therefore\ C = \pi D, \text{ where } D \text{ is the diameter,}$$

$$\text{or } C = 2\pi r, \text{ where } r \text{ is the radius.}$$

Ex. 1. Find the circumference of a circle of radius 21 inches.

$$C = 2\pi r$$

$$= 2 \times \frac{22}{\cancel{7}} \times \overset{3}{\cancel{21}} \text{ inches}$$

$$= 132 \text{ sq. in.}$$

Ex. 2. Find the radius of a circle of circumference 45 ft. 9 in., using $\pi = 3{\cdot}142$.

$$C = 2\pi r.$$

$$\therefore\ r = \frac{c}{2\pi}$$

$$= \frac{45{\cdot}75}{2 \times 3{\cdot}142},$$

$$\text{Log } 45{\cdot}75 = 1{\cdot}6604$$

$$\text{Log } 6{\cdot}284 = \ {\cdot}7983$$

$$\text{Log } 45{\cdot}75 - \text{Log } 6{\cdot}284 = {\cdot}8621 = \log r.$$

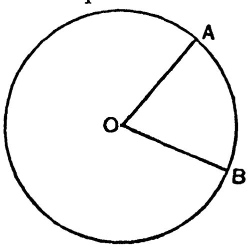
$$\therefore\ r = 7{\cdot}28 \text{ ft.}$$

or $45{\cdot}75 \div 6{\cdot}284$

```
        7·280
6·284)45·750
      43·988
       17620
       12568
        50520
        50272
         2480.
```

62. The Meaning of a Radian.

Imagine an arc AB which, if stretched out into a straight line, would be found to be equal to the radius.



The angle between the two radii OA and OB is known as a *radian*.

We know that the circumference $= 2\pi r$. Also AB has the same ratio to the circumference as the angle AOB has to four right angles.

$$\therefore\ \frac{\text{arc } AB}{\text{circumference}} = \frac{\angle AOB}{4 \text{ rt. } \angle\text{s}} = \frac{1 \text{ radian}}{4 \text{ rt. } \angle\text{s}}.$$

or

$$\frac{r}{2\pi r} = \frac{1 \text{ radian}}{4 \text{ rt. } \angle\text{s}}.$$

$$\therefore\ \frac{1}{2\pi} = \frac{1 \text{ radian}}{4 \text{ rt. } \angle\text{s}}$$

or

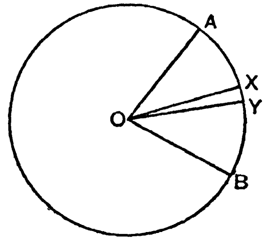
$$\frac{1}{\pi} = \frac{1 \text{ radian}}{2 \text{ rt. } \angle\text{s}}.$$

$$\therefore \pi \text{ radians} = 180° \text{ or } 2 \text{ rt. } \angle\text{s}.$$

$$\therefore 1 \text{ radian} = \frac{180}{3{\cdot}1416}$$

$$= 57{\cdot}29° \text{ approximately.}$$

63. The Area of a Circle.



Let $\angle AOB$ be one radian.

Imagine this split up into a very large number of triangles, such as OXY, in which OX and OY are radii very close together.

The area of OXY is almost exactly equal to $\frac{1}{2}XY$ times the radius, and if all the triangles making up the sector AOB are added together, we get half the arc AB times the radius as the area.

$$\therefore \text{sector } AOB = \tfrac{1}{2}r \times r, \text{ for the arc } AB = r,$$

$$= \tfrac{1}{2}r^2.$$

But $$\frac{\text{sector } AOB}{\text{circle}} = \frac{\angle AOB}{4 \text{ rt. } \angle\text{s}} = \frac{1 \text{ radian}}{2\pi \text{ radians}} = \frac{1}{2\pi}.$$

$$\therefore \frac{\text{circle}}{\text{sector } AOB} = \frac{2\pi}{1}.$$

$$\therefore \text{circle} = 2\pi \times \text{sector } AOB$$

$$= 2\pi \times \tfrac{1}{2}r^2$$

$$= \pi r^2.$$

Learn:

The area of a circle is πr^2, *where* r *is the radius and* $\pi =$ *approximately* $3\frac{1}{7}$ *or* $3{\cdot}1416$.

Ex. 1. Find the area of a circle of radius 7 inches.

$$A = \pi r^2 = \tfrac{22}{7} \times 7 \text{ sq. in.} = 22 \text{ sq. in.}$$

Ex. 2. Find the diameter of a circle of area 456 sq. in.

$$A = \pi r^2; \; r^2 = \frac{A}{\pi}; \; r = \sqrt{\frac{A}{\pi}}; \; d = 2\sqrt{\frac{A}{\pi}}.$$

$$\therefore d = 2\sqrt{\frac{456}{\pi}} \text{ inches.}$$

The result can be found by arithmetical computation or by using logs.

EXERCISE 32.

Find the circumferences of the following circles, using $\pi = \frac{22}{7}$.

1. $r = 14''$. 2. $r = 4{\cdot}2''$. 3. $D = 14$ ft.

4. $D = 32$ ft. 5. $r = 15$ ft. 6 in. 6. $D = 13{\cdot}12$ in.

Find the circumferences of the following circles, using $\pi = 3{\cdot}1416$ and giving answers correct to the nearest hundredth of the unit:

7. $r = 15''$. 8. $r = 1{\cdot}85''$. 9. $D = 73$ ft.

Find the diameter and radius in each of the following circles, the area being given in each case, using $\pi = 3{\cdot}142$.

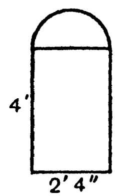
10. 25 sq. ft. 11. 19 sq. ft. 12. 1 sq. ft. 3 sq. in.

13. 4·73 sq. in. 14. ·159 sq. ft. 15. ·087 sq. in.

16. A right-angled triangle has two sides 5 and 12 inches. What is the area of the semi-circle described on the hypotenuse as diameter?

17. Construct a circle which has an area half that of a circle with diameter $3\frac{1}{2}$ in.

18. Find the area and weight of the piece of steel plate shown in the sketch. The upper portion is semi-circular, and one square foot of the plate weighs 5·1 lb.



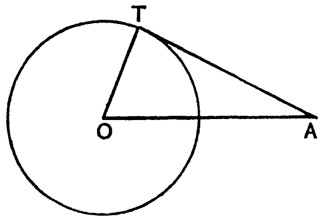
19. Draw a right-angled isosceles triangle ABC whose hypotenuse AB is 12 cm. Bisect the hypotenuse in D, and draw a circle passing through B, C and D. Calculate the area of the circle.

20. Find the radius of a circle which is equal in area to the sum of the areas of two circles of radii 2 ft. and 5 ft. 10 in. respectively.

21. Draw an equilateral triangle with base $1\frac{1}{2}''$. On each side as diameter, draw a semi-circle outside the triangle. Calculate the distance round the outside of the figure so obtained.

GENERAL REVISION

EXERCISE 33 (*a*). MENTAL.

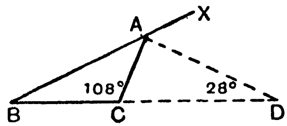


1. If AT is a tangent to the circle and $\angle TAO = 27°$, what is $\angle TOA$?

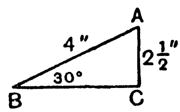
2. If in the figure of Question 1, $OT = 2\frac{1}{2}''$ and $OA = 6\frac{1}{2}''$, find AT.

3. If the extremities of all tangents 1″ long drawn to a given circle were joined, what figure would be formed?

4. If a parallelogram can be inscribed in a circle, why must it be a rectangle?



5. AD bisects the exterior angle CAX of the $\triangle ABC$. If $\angle ACB = 108°$ and $\angle ADC = 28°$, what is $\angle BAC$?



6. How many different triangles can be made with the dimensions shown in the rough diagram?

7. The angles of a triangle are in the ratio $4:5:6$. Find them.

8. If the sum of two opposite angles of a parallelogram is 240°, what is each of the other two angles?

EXERCISE 33 (*b*).

1. Find the number of degrees in each angle of a regular hexagon $ABCDEF$. Prove that ACE is an equilateral triangle.

2. Draw a straight line AB 3 inches long, and take a point O in AB 1 inch from A. Without using a protractor or set-square draw through O a straight line OC perpendicular to AB and 2 inches long. Join AC and BC. Measure the length of AC and the angle ABC.

3. A man stands at the centre of the top of a building which is in the form of a cylinder 20 feet high and of radius 10 feet. His eyes are 5 feet above the top of the tower. Draw a figure, taking a length of 1 inch to represent 10 feet, and hence find the least possible distance from the base of the cylinder to the nearest points in the ground which he can see.

4. A is a point outside a given straight line XY. B is a point on the line XY. Find a point on the line XY which is equidistant from A and B. State and prove your construction.

5. Construct a quadrilateral $ABCD$ in which the diagonal AC is 7·7 cm. long, BD cuts AC at O so that AO is 3·3 cm., $AB = 4{\cdot}4$ cm., $\angle AOB = 47°$, and $\angle BCD = 59°$.

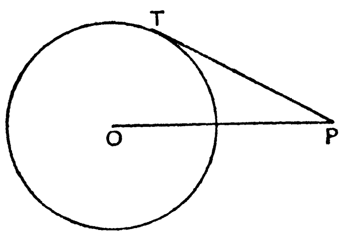
6. Draw a rectangle $ABCD$, in which $AB = 10$ cm. and the diagonal $AC = 11$ cm. Within this rectangle construct a crescent composed of two circular arcs, given that the crescent touches CD, has its horns at A and B, and is of maximum thickness 2 cm.

7. Show how to construct a parallelogram, given the length of each diagonal and the sum of the angles which one side subtends at the other two vertices.

8. $ABCD$ is a quadrilateral in which $AB = 3{\cdot}12$ in., $BC = 2{\cdot}34$ in., $CD = 3{\cdot}60$ in., $DA = 1{\cdot}50$ in., and the diagonal $AC = 3{\cdot}90$ in. Prove that two of the angles of the quadrilateral are right angles, and find its area in square inches.

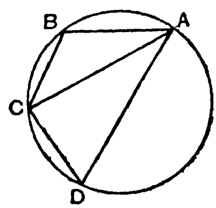
EXERCISE 34 (*a*). MENTAL.

1. PT is a tangent 12 cm. long and $OP = 13$ cm. Find the diameter of the circle.



2. Give a formula for the length of a tangent to a circle of radius r inches from a point c inches from the centre.

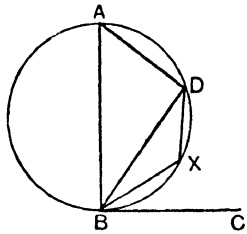
3. If $\angle BAD$ is bisected by AC, how do you know that $BC = CD$?



4. What angle is subtended at the centre of a circle by a side of a regular 12-sided figure inscribed in the circle?

5. Two angles of a triangle are 28° and 108°. What are the sizes of the three exterior angles?

6. BC is a tangent to a circle and AB is a diameter. How do we know that $\angle DBC = \angle BAD$?



7. If in this figure $\angle DBC = 30°$, what is the angle DXB, and why?

8. What is the area of a triangle with sides 5, 12 and 13 ins.?

EXERCISE 34 (*b*).

1. Construct a parallelogram $ABCD$ having AB 2·5 in., angle ABC 50°, and the area equal to the area of a rectangle whose adjacent sides are 3 in. and 2 in. long. Describe your construction.

2. The perpendicular bisector of the side BC of a triangle ABC meets the circumference of the circumscribed circle in D on the side remote from A. Prove that AD bisects the angle BAC.

3. Draw a circle of radius 5·3 cm., and take a point P at a distance of 11 cm. from its centre. Construct the tangents from P to the circle, and measure the angle between them.

4. On a line AB of length 4 inches as base, construct a triangle ABC having its area 6 square inches and AC $3\frac{1}{2}$ inches long. Give reasons for your construction.

5. A chord AB of a circle subtends an angle of 60° at the centre O. AO meets the circle again at D and C is a point in AB produced such that $AB = BC$. Prove that the circle on CD as diameter passes through O.

6. A is a point 300 ft. due west of a flagstaff 100 ft. high. Find by careful drawings to a scale of 2 in. = 100 ft. the distance from the foot of the flagstaff of a point B 200 ft. south-east of A, and the angle of elevation of the top of the flagstaff when viewed from B.

7. Without using a set-square construct a triangle ABC with the angle B a right angle, AB 2 inches long and BC 1·4 inches long.

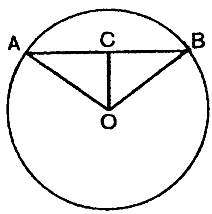
Through C draw a straight line to pass between the points A and B and be equidistant from them.

Measure the distance of A or B from the straight line which you have drawn.

8. Draw two triangles in each of which two sides are 7 cm. and 4·8 cm. in length, and the angle opposite the shorter side is 35°. Show that the sum of the angles opposite the greatest sides of these two triangles is two right angles.

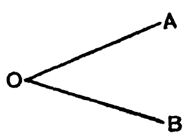
EXERCISE 35 (*a*). MENTAL.

1. O is the centre of the circle and OC is perpendicular to AB. If $OB = 5$ cm. and $OC = 3$ cm., what is AB?

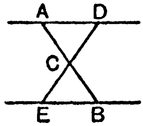


2. Find the area of the triangle OAB.

3. If all possible circles were drawn touching OA and OB, what line do you think would be formed by joining the centres?



4. AD, EB are parallel lines and C is the middle point of AB. How do you know that $EC = CD$?



5. How many circles in all can be drawn touching the three sides of a triangle, or one of the sides and the other two produced?

6. What is the length of the diagonal of a rectangle if the sides are 5″ and 12″?

7. If I go N. 10 miles, then E. 3 miles, then S. 2 miles and finally W. 8 miles, how far N. and W. of my starting-point am I then?

8. The diagonals of a rhombus are 8″ and 5″. What is the area?

EXERCISE 35 (*b*).

1. AB, BC, CD are three consecutive sides of a regular five-sided figure. AB and DC when produced meet at E. Calculate the number of degrees in the angle AED.

2. Construct a parallelogram with adjacent sides 5·9 cm. and 4·8 cm. long, and including an angle of 40°. Construct a triangle of the same area as this parallelogram.

3. Two lines AB and AC, each 5·6 cm. long, are at right angles to each other. Find two points, each equidistant from A and B and 3·4 cm. from C.

4. A cone, the radius of whose base is 7 cm. and whose height is 15 cm., lies on its side on a horizontal plane. Find from a figure the distance of the highest point of the cone from the plane.

5. Draw two lines OA and OB each 3 inches long, including an angle of 70°. From A, and on the same side of OA as B, draw a straight line AP such that the angle OAP is 100°. Find two points in AP each $2\frac{3}{4}$ inches from B.

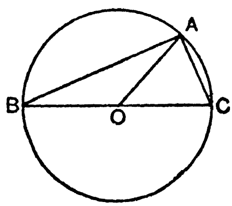
6. A, B are two points one mile and a half apart on a straight coast-line and A is north-east of B. An observer at A sees a ship at sea 30° west of north, and at the same instant an observer at B sees the ship 20° east of north. Find, by careful drawing to a scale of 4 inches to the mile, the distance of the ship from the coast.

7. Construct a trapezium $ABCD$ in which the parallel sides AB, CD are 3 inches long and 5 inches long respectively, and the slant sides BC, AD are each 2 inches long. Prove that the area of the trapezium is four times the area of an equilateral triangle whose sides are 2 inches long.

8. Calculate (correct to one-tenth of an inch) the length of each side of a right-angled isosceles triangle whose area is 8 square inches.

EXERCISE 36 (*a*). MENTAL.

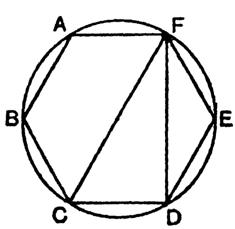
1. If BC is a diameter of a circle and $\angle ABO = 20°$, find $\angle AOC$ and $\angle OAC$.



2. Give the value of π to 4 places of decimals.

3. How do you describe a circle circumscribing a triangle?

4. $ABCDEF$ is a regular hexagon described in a circle. What is the size of the angle CFD, and why?



5. If two triangles are such that the sum of the base angles of one is equal to the sum of the base angles of the other, why are the vertical angles equal?

6. Three angles of a quadrilateral are 84°, 85°, 86°. What is the fourth?

7. If in the figure of Question 4 $CD = 1''$, what is CF?

8. How many congruent triangles can be made in different positions on a given base, if the three sides are given equal in each?

EXERCISE 36 (*b*).

Being the paper set in December 1926 *at the Junior Local Examination of the University of Oxford.*

1. Prove that if the sides of a convex polygon are produced in order, the sum of the angles thus formed is equal to four right angles.

Three angles of a hexagon are right angles and the other three angles are equal to each other. Find the number of degrees in each of the equal angles.

2. Prove that the parallelogram whose diagonals are at right angles has all its sides equal (i.e. is a rhombus).

Construct a rhombus with a diagonal 6 in. long and an area of 10 sq. in. Explain your construction.

3. The hypotenuse of a right-angled triangle is 8 in. and one side is 3 in. long. Calculate the length of the other side to the nearest hundredth of an inch, and the area of the triangle to the nearest tenth of a square inch.

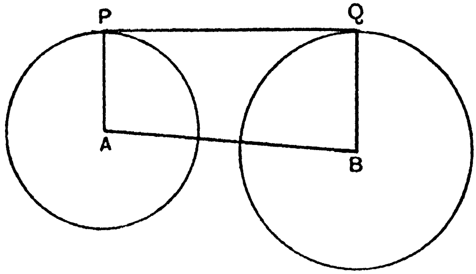
4. AB, CD are two chords of a circle which are parallel to one another. Prove that the triangles ACD, BDC are equal in all respects.

5. Construct a triangle ABC in which $AB = 4$ in., $AC = 5$ in., and $BC = 6$ in. Construct the perpendicular AD from A to BC and draw the circles circumscribing the triangles ADB, ADC.

6. The sloping faces of a square pyramid standing on a base 3 in. square are all equilateral triangles. Find, by calculation, or by careful drawing, the height of the vertex of the pyramid above the base.

EXERCISE 37 (*a*). MENTAL.

1. If PQ is a common tangent to the two circles, why is PA parallel to QB?

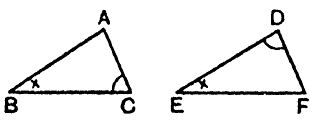


2. If all the parallel chords of a circle are bisected, what line would be formed by joining up all the middle points?

3. Give a formula connecting the length of the direct common tangents to two circles, the radii and the distance between the centres.

4. If two radii of a circle contain an angle of 60°, what is the length of the chord joining the extremities?

5. If $\angle ABC = \angle DEF$ and $\angle ACB = \angle EDF$, what else must we know before the triangles can be said to be congruent?



6. What is the area of an equilateral triangle with a side 2 in.?

7. If two tangents to a circle from an external point contain an angle of 60° and the radius of the circle is 3″, what is the distance from the point to the centre?

8. The area of a triangle is 10 sq. cm. and the height is 5 cm. What is the length of the base?

EXERCISE 37 (*b*).

Being the whole of Part I of the paper set in December 1926 *at the Junior Local Examination of the University of Cambridge. In this Examination candidates can pass by doing sufficiently well in Part I.*

PART I.

1. Draw a square $ABCD$ of side 2 inches, and on the side AD construct an equilateral triangle ADE outside the square. Construct a triangle of area equal to that of the figure $ABCDE$ with one vertex at E and with its base on the line of which BC forms a part. Measure the base of this triangle.

2. Prove that the sum of the angles of a triangle is equal to two right angles.

Show that two triangles can be drawn in each of which two sides are 8 cm. and 5·5 cm. in length, and the angle opposite the shorter side is 42°. Show that the sum of the angles opposite the greatest sides of these two triangles is two right angles.

3. Prove that the angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

O is the centre of the circle circumscribing the triangle ABC; AO, BO, CO are drawn, and CO and BO are produced to meet AB and AC respectively in F and E. Prove that the sum of the angles BFC, BEC is equal to three times the angle BAC.

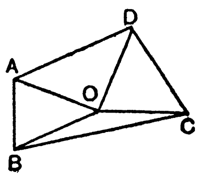
4. Prove that in a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the sides containing the right angle.

P is a point outside a rectangle $ABCD$ (for convenience P may be taken above and to the right of the rectangle); prove that

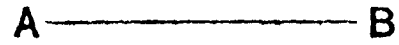
$$PA^2 + PC^2 = PB^2 + PD^2.$$

EXERCISE 38 (*a*). MENTAL.

1. If $\angle BAD = 108°$ and O is the centre of a circle passing through A, B, C and D, calculate $\angle BOD$ and $\angle BCD$.

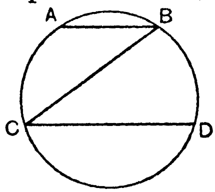


2. If all possible circles were described passing through A and B, what line would be formed by joining the centres?



3. When is it impossible to construct a circle to pass through two given points and have its centre on a given line?

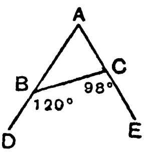
4. If AB and CD are parallel chords, how do you know that $AC = BD$?



5. If I go 8 miles due N. and then 6 miles due W., how far am I from my starting-point?

6. The angles at the base of an isosceles triangle are each twice the vertical angle. What is the size of the latter?

7. The exterior angles DBC and BCE of the triangle ABC are 120° and 98°. What is $\angle BAC$?

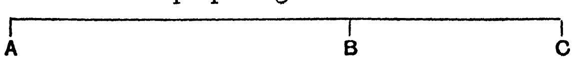


8. The distance from the centre of a circle to each of 6 equal chords forming a hexagon is 1·73″ and the chords are 2″ in length. What is the area of the hexagon?

EXERCISE 38 (*b*).

Being the Geometry Questions of the 1927 *Junior School Commercial Certificate of the Royal Society of Arts.*

1. Distances measured from left to right, such as AB on the line below, are positive; distances measured from right to left, such as BA, are negative. Find in inches as accurately as you can the distances AB, BC, CA, and write them down in your answer book with the proper sign.



What is the magnitude of $AB + BC + CA$? If the length BC is called x, what is CA in terms of x?

2. Prove that if O is the centre of a circle, BC is any chord and K is the middle point of the chord, then OK is perpendicular to BC.

3. Prove that if the angle S of the triangle RST is a right angle, then $RT^2 = RS^2 + ST^2$.

The wall of a house is 25 feet high. Find, either by calculation or by drawing and measurement, the length of the shortest ladder that will reach to the top of the wall, if the foot of the ladder cannot stand less than 15 feet from the foot of the wall.

4. Draw a triangle ABC having the side BC 9·4 centimetres long, the angle ABC equal to 62° and the angle ACB equal to 53°.

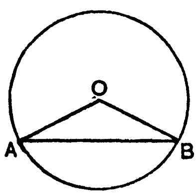
By a geometrical construction draw a line bisecting the angle ABC and prove that your construction is correct.

The bisector of the angle ABC meets AC in E. Work out in degrees the sizes of the angles BAC, AEB, CEB. Check your results by measurement with a protractor and write your measurements in your answer book.

Note. *These were four of the six questions, for which two hours were allowed, the other two questions being purely Algebraical.*

EXERCISE 39 (*a*). MENTAL.

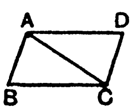
1. If $\angle OAB = 28°$, find $\angle AOB$.



2. What is the size of each angle of a regular polygon of 12 sides?

3. If the middle points of all equal chords of a circle were joined, what figure do you think would be formed?

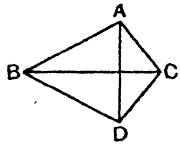
4. If $AB = CD$ and $AD = BC$, why is the quadrilateral $ABCD$ bound to be a parallelogram?



5. If a rhombus and a square have equal sides, which has the greater area and why?

6. If the angle $(2x - 45)°$ is the complement of $(90 - x)°$, find x.

7. If $AB = BD$ and $AC = CD$, how do you know that $\angle BAC = \angle BDC$?



8. If in the above figure $AC = 3$, $AB = 4$, $BC = 5$, what is the area of the quadrilateral?

EXERCISE 39 (b).

Being Section III of the East Midland Educational Union Central Schools Examination 1927.

1. Draw a line $OP = 1\frac{1}{4}$ inches. On OP as diameter describe a circle. With centre O and radius OP describe a circle. In the circumference of the larger circle take any point T. Join PT cutting the circumference of the smaller circle at Q. Join OQ. What do you discover by measuring angle OQP, and lines TQ, QP?

Give reasons for each result.

2. Construct a right-angled triangle ABC. Make the hypotenuse $AC = 6{\cdot}8$ cm.; make one side $AB = 3$ cm. Join the point B to the middle point of AC. Measure the angle at A with protractor. Also measure the line from B to the middle point of AC, give its length in centimetres and mention anything you discover respecting it.

3. On AB, a line $1\frac{1}{2}$ inches long, describe an equilateral triangle ABC. In AB take any point D; join CD.

Prove that the angle BDC equals the sum of the angles DAC and ACD.

SINES

	0′	10′	20′	30′	40′	50′	60′		Differences. 1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	0·0000	0·0029	0·0058	0·0087	0·0116	0·0145	0·0175	89°	3	6	9	12	15	17	20	23	26
1	·0175	·0204	·0233	·0262	·0291	·0320	·0349	88	3	6	9	12	15	17	20	23	26
2	·0349	·0378	·0407	·0436	·0465	·0494	·0523	87	3	6	9	12	15	17	20	23	26
3	·0523	·0552	·0581	·0610	·0640	·0669	·0698	86	3	6	9	12	15	17	20	23	26
4	·0698	·0727	·0756	·0785	·0814	·0843	·0872	85	3	6	9	12	15	17	20	23	26
5	0·0872	0·0901	0·0929	0·0958	0·0987	0·1016	0·1045	84	3	6	9	12	14	17	20	23	26
6	·1045	·1074	·1103	·1132	·1161	·1190	·1219	83	3	6	9	12	14	17	20	23	26
7	·1219	·1248	·1276	·1305	·1334	·1363	·1392	82	3	6	9	12	14	17	20	23	26
8	·1392	·1421	·1449	·1478	·1507	·1536	·1564	81	3	6	9	11	14	17	20	23	26
9	·1564	·1593	·1622	·1650	·1679	·1708	·1736	80°	3	6	9	11	14	17	20	23	26
10°	0·1736	0·1765	0·1794	0·1822	0·1851	0·1880	0·1908	79	3	6	9	11	14	17	20	23	26
11	·1908	·1937	·1965	·1994	·2022	·2051	·2079	78	3	6	9	11	14	17	20	23	26
12	·2079	·2108	·2136	·2164	·2193	·2221	·2250	77	3	6	9	11	14	17	20	23	26
13	·2250	·2278	·2306	·2334	·2363	·2391	·2419	76	3	6	8	11	14	17	20	23	25
14	·2419	·2447	·2476	·2504	·2532	·2560	·2588	75	3	6	8	11	14	17	20	23	25
15	0·2588	0·2616	0·2644	0·2672	0·2700	0·2728	0·2756	74	3	6	8	11	14	17	20	22	25
16	·2756	·2784	·2812	·2840	·2868	·2896	·2924	73	3	6	8	11	14	17	20	22	25
17	·2924	·2952	·2979	·3007	·3035	·3062	·3090	72	3	6	8	11	14	17	19	22	25
18	·3090	·3118	·3145	·3173	·3201	·3228	·3256	71	3	6	8	11	14	17	19	22	25
19	·3256	·3283	·3311	·3338	·3365	·3393	·3420	70°	3	5	8	11	14	16	19	22	25
20°	0·3420	0·3448	0·3475	0·3502	0·3529	0·3557	0·3584	69	3	5	8	11	14	16	19	22	25
21	·3584	·3611	·3638	·3665	·3692	·3719	·3746	68	3	5	8	11	14	16	19	22	24
22	·3746	·3773	·3800	·3827	·3854	·3881	·3907	67	3	5	8	11	13	16	19	21	24
23	·3907	·3934	·3961	·3987	·4014	·4041	·4067	66	3	5	8	11	13	16	19	21	24
24	·4067	·4094	·4120	·4147	·4173	·4200	·4226	65	3	5	8	11	13	16	19	21	24
25	0·4226	0·4253	0·4279	0·4305	0·4331	0·4358	0·4384	64	3	5	8	10	13	16	18	21	24
26	·4384	·4410	·4436	·4462	·4488	·4514	·4540	63	3	5	8	10	13	16	18	21	23
27	·4540	·4566	·4592	·4617	·4643	·4669	·4695	62	3	5	8	10	13	15	18	21	23
28	·4695	·4720	·4746	·4772	·4797	·4823	·4848	61	3	5	8	10	13	15	18	20	23
29	·4848	·4874	·4899	·4924	·4950	·4975	·5000	60°	3	5	8	10	13	15	18	20	23
30°	0·5000	0·5025	0·5050	0·5075	0·5100	0·5125	0·5150	59	3	5	8	10	13	15	18	20	23
31	·5150	·5175	·5200	·5225	·5250	·5275	·5299	58	2	5	7	10	12	15	17	20	22
32	·5299	·5324	·5348	·5373	·5398	·5422	·5446	57	2	5	7	10	12	15	17	20	22
33	·5446	·5471	·5495	·5519	·5544	·5568	·5592	56	2	5	7	10	12	15	17	19	22
34	·5592	·5616	·5640	·5664	·5688	·5712	·5736	55	2	5	7	10	12	14	17	19	22
35	0·5736	0·5760	0·5783	0·5807	0·5831	0·5854	0·5878	54	2	5	7	9	12	14	17	19	21
36	·5878	·5901	·5925	·5948	·5972	·5995	·6018	53	2	5	7	9	12	14	16	19	21
37	·6018	·6041	·6065	·6088	·6111	·6134	·6157	52	2	5	7	9	12	14	16	18	21
38	·6157	·6180	·6202	·6225	·6248	·6271	·6293	51	2	5	7	9	11	14	16	18	20
39	·6293	·6316	·6338	·6361	·6383	·6406	·6428	50°	2	4	7	9	11	13	16	18	20
40°	0·6428	0·6450	0·6472	0·6494	0·6517	0·6539	0·6561	49	2	4	7	9	11	13	15	18	20
41	·6561	·6583	·6604	·6626	·6648	·6670	·6691	48	2	4	7	9	11	13	15	17	20
42	·6691	·6713	·6734	·6756	·6777	·6799	·6820	47	2	4	6	9	11	13	15	17	19
43	·6820	·6841	·6862	·6884	·6905	·6926	·6947	46	2	4	6	8	11	13	15	17	19
44	·6947	·6967	·6988	·7009	·7030	·7050	·7071	45	2	4	6	8	10	12	15	17	19
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

COSINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
45°	0·7071	0·7092	0·7112	0·7133	0·7153	0·7173	0·7193	44°	2	4	6	8	10	12	14	16	18
46	·7193	·7214	·7234	·7254	·7274	·7294	·7314	43	2	4	6	8	10	12	14	16	18
47	·7314	·7333	·7353	·7373	·7392	·7412	·7431	42	2	4	6	8	10	12	14	16	18
48	·7431	·7451	·7470	·7490	·7509	·7528	·7547	41	2	4	6	8	10	12	13	15	17
49	·7547	·7566	·7585	·7604	·7623	·7642	·7660	40°	2	4	6	8	9	11	13	15	17
50°	0·7660	0·7679	0·7698	0·7716	0·7735	0·7753	0·7771	39	2	4	6	7	9	11	13	15	17
51	·7771	·7790	·7808	·7826	·7844	·7862	·7880	38	2	4	5	7	9	11	13	14	16
52	·7880	·7898	·7916	·7934	·7951	·7969	·7986	37	2	4	5	7	9	11	12	14	16
53	·7986	·8004	·8021	·8039	·8056	·8073	·8090	36	2	3	5	7	9	10	12	14	16
54	·8090	·8107	·8124	·8141	·8158	·8175	·8192	35	2	3	5	7	8	10	12	14	15
55	0·8192	0·8208	0·8225	0·8241	0·8258	0·8274	0·8290	34	2	3	5	7	8	10	12	13	15
56	·8290	·8307	·8323	·8339	·8355	·8371	·8387	33	2	3	5	6	8	10	11	13	14
57	·8387	·8403	·8418	·8434	·8450	·8465	·8480	32	2	3	5	6	8	9	11	13	14
58	·8480	·8496	·8511	·8526	·8542	·8557	·8572	31	2	3	5	6	8	9	11	12	14
59	·8572	·8587	·8601	·8616	·8631	·8646	·8660	30°	1	3	4	6	7	9	10	12	13
60°	0·8660	0·8675	0·8689	0·8704	0·8718	0·8732	0·8746	29	1	3	4	6	7	9	10	11	13
61	·8746	·8760	·8774	·8788	·8802	·8816	·8829	28	1	3	4	6	7	8	10	11	12
62	·8829	·8843	·8857	·8870	·8884	·8897	·8910	27	1	3	4	5	7	8	9	11	12
63	·8910	·8923	·8936	·8949	·8962	·8975	·8988	26	1	3	4	5	6	8	9	10	12
64	·8988	·9001	·9013	·9026	·9038	·9051	·9063	25	1	3	4	5	6	8	9	10	11
65	0·9063	0·9075	0·9088	0·9100	0·9112	0·9124	0·9135	24	1	2	4	5	6	7	8	10	11
66	·9135	·9147	·9159	·9171	·9182	·9194	·9205	23	1	2	3	5	6	7	8	9	10
67	·9205	·9216	·9228	·9239	·9250	·9261	·9272	22	1	2	3	4	6	7	8	9	10
68	·9272	·9283	·9293	·9304	·9315	·9325	·9336	21	1	2	3	4	5	6	7	9	10
69	·9336	·9346	·9356	·9367	·9377	·9387	·9397	20°	1	2	3	4	5	6	7	8	9
70°	0·9397	0·9407	0·9417	0·9426	0·9436	0·9446	0·9455	19	1	2	3	4	5	6	7	8	9
71	·9455	·9465	·9474	·9483	·9492	·9502	·9511	18	1	2	3	4	5	6	6	7	8
72	·9511	·9520	·9528	·9537	·9546	·9555	·9563	17	1	2	3	4	4	5	6	7	8
73	·9563	·9572	·9580	·9588	·9596	·9605	·9613	16	1	2	2	3	4	5	6	7	7
74	·9613	·9621	·9628	·9636	·9644	·9652	·9659	15	1	2	2	3	4	5	5	6	7
75	0·9659	0·9667	0·9674	0·9681	0·9689	0·9696	0·9703	14	1	1	2	3	4	4	5	6	7
76	·9703	·9710	·9717	·9724	·9730	·9737	·9744	13	1	1	2	3	3	4	5	5	6
77	·9744	·9750	·9757	·9763	·9769	·9775	·9781	12	1	1	2	3	3	4	4	5	6
78	·9781	·9787	·9793	·9799	·9805	·9811	·9816	11	1	1	2	2	3	3	4	5	5
79	·9816	·9822	·9827	·9833	·9838	·9843	·9848	10°	1	1	2	2	3	3	4	4	5
80°	0·9848	0·9853	0·9858	0·9863	0·9868	0·9872	0·9877	9	0	1	1	2	2	3	3	4	4
81	·9877	·9881	·9886	·9890	·9894	·9899	·9903	8	0	1	1	2	2	3	3	3	4
82	·9903	·9907	·9911	·9914	·9918	·9922	·9925	7	0	1	1	2	2	2	3	3	3
83	·9925	·9929	·9932	·9936	·9939	·9942	·9945	6	0	1	1	1	2	2	2	3	3
84	·9945	·9948	·9951	·9954	·9957	·9959	·9962	5	0	1	1	1	1	2	2	2	3
85	0·9962	0·9964	0·9967	0·9969	0·9971	0·9974	0·9976	4									
86	·9976	·9978	·9980	·9981	·9983	·9985	·9986	3									
87	·9986	·9988	·9989	·9990	·9992	·9993	·9994	2									
88	·9994	·9995	·9996	·9997	·9997	·9998	·9998	1									
89	0·9998	0·9999	0·9999	1·0000	1·0000	1·0000	1·0000	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

Differences are so small here that tabulation is unnecessary.

COSINES

									Differences.								
	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
0°	0·0000	0·0029	0·0058	0·0087	0·0116	0·0145	0·0175	89°	3	6	9	12	15	17	20	23	26
1	·0175	·0204	·0233	·0262	·0291	·0320	·0349	88	3	6	9	12	·5	17	20	23	26
2	·0349	·0378	·0407	·0437	·0466	·0495	·0524	87	3	6	9	12	15	18	20	23	26
3	·0524	·0553	·0582	·0612	·0641	·0670	·0699	86	3	6	9	12	15	18	20	23	26
4	·0699	·0729	·0758	·0787	·0816	·0846	·0875	85	3	6	9	12	15	18	21	23	26
5	0·0875	0·0904	0·0934	0.0963	0·0992	0·1022	0·1051	84	3	6	9	12	15	18	21	24	26
6	·1051	·1080	·1110	·1139	·1169	·1198	·1228	83	3	6	9	12	15	18	21	24	27
7	·1228	·1257	·1287	·1317	·1346	·1376	·1405	82	3	6	9	12	15	18	21	24	27
8	·1405	·1435	·1465	·1495	·1524	·1554	·1584	81	3	6	9	12	15	18	21	24	27
9	·1584	·1614	·1644	·1673	·1703	·1733	·1763	80°	3	6	9	12	15	18	21	24	27
10°	0·1763	0·1793	0·1823	0·1853	0·1883	0·1914	0·1944	79	3	6	9	12	15	18	21	24	27
11	·1944	·1974	·2004	·2035	·2065	·2095	·2126	78	3	6	9	12	15	18	21	24	27
12	·2126	·2156	·2186	·2217	·2247	·2278	·2309	77	3	6	9	12	15	18	21	24	27
13	·2309	·2339	·2370	·2401	·2432	·2462	·2493	76	3	6	9	12	15	18	22	25	28
14	·2493	·2524	·2555	·2586	·2617	·2648	·2679	75	3	6	9	12	16	19	22	25	28
15	0·2679	0·2711	0·2742	0·2773	0·2805	0·2836	0·2867	74	3	6	9	13	16	19	22	25	28
16	·2867	·2899	·2931	·2962	·2994	·3026	·3057	73	3	6	9	13	16	19	22	25	28
17	·3057	·3089	·3121	·3153	·3185	·3217	·3249	72	3	6	10	13	16	19	22	26	29
18	·3249	·3281	·3314	·3346	·3378	·3411	·3443	71	3	6	10	13	16	19	23	26	29
19	·3443	·3476	·3508	·3541	·3574	·3607	·3640	70°	3	7	10	13	16	20	23	26	29
20°	0·3640	0·3673	0·3706	0·3739	0·3772	0·3805	0·3839	69	3	7	10	13	17	20	23	27	30
21	·3839	·3872	·3906	·3939	·3973	·4006	·4040	68	3	7	10	13	17	20	24	27	30
22	·4040	·4074	·4108	·4142	·4176	·4210	·4245	67	3	7	10	14	17	20	24	27	31
23	·4245	·4279	·4314	·4348	·4383	·4417	·4452	66	3	7	10	14	17	21	24	28	31
24	·4452	·4487	·4522	·4557	·4592	·4628	·4663	65	4	7	11	14	18	21	25	28	32
25	0·4663	0·4699	0.4734	0·4770	0·4806	0·4841	0·4877	64	4	7	11	14	18	21	25	29	32
26	·4877	·4913	·4950	·4986	·5022	·5059	·5095	63	4	7	11	15	18	22	25	29	33
27	·5095	·5132	·5169	·5206	·5243	·5280	·5317	62	4	7	11	15	18	22	26	30	33
28	·5317	·5354	·5392	·5430	·5467	·5505	·5543	61	4	8	11	15	19	23	26	30	34
29	·5543	·5581	·5619	·5658	·5696	·5735	·5774	60°	4	8	12	15	19	23	27	31	35
30°	0·5774	0·5812	0·5851	0·5890	0·5930	0·5969	0·6009	59	4	8	12	16	20	24	27	31	35
31	·6009	·6048	·6088	·6128	·6168	·6208	·6249	58	4	8	12	16	20	24	28	32	36
32	·6249	·6289	·6330	·6371	·6412	·6453	·6494	57	4	8	12	16	20	25	29	33	37
33	·6494	·6536	·6577	·6619	·6661	·6703	·6745	56	4	8	13	17	21	25	29	33	38
34	·6745	·6787	·6830	·6873	·6916	·6959	·7002	55	4	9	13	17	21	26	30	34	39
35	0·7002	0·7046	0·7089	0·7133	0·7177	0·7221	0·7265	54	4	9	13	18	22	26	31	35	40
36	·7265	·7310	·7355	·7400	·7445	·7490	·7536	53	5	9	14	18	23	27	32	36	41
37	·7536	·7581	·7627	·7673	·7720	·7766	·7813	52	5	9	14	18	23	28	32	37	42
38	·7813	·7860	·7907	·7954	·8002	·8050	·8098	51	5	10	14	19	24	29	33	38	43
39	·8098	·8146	·8195	·8243	·8292	·8342	·8391	50°	5	10	15	20	24	29	34	39	44
40°	0·8391	0·8441	0·8491	0·8541	0·8591	0·8642	0·8693	49	5	10	15	20	25	30	35	40	45
41	·8693	·8744	·8796	·8847	·8899	·8952	·9004	48	5	10	16	21	26	31	36	41	47
42	·9004	·9057	·9110	·9163	·9217	·9271	·9325	47	5	11	16	21	27	32	37	43	48
43	·9325	·9380	·9435	·9490	·9545	·9601	·9657	46	6	11	17	22	28	33	39	44	50
44	·9657	·9713	·9770	·9827	·9884	·9942	1·0000	45	6	11	17	23	29	34	40	46	51
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

COTANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Differences.								
45°	1·0000	1·0058	1·0117	1·0176	1·0235	1·0295	1·0355	44°	6	12	18	24	30	36	41	47	53
46	·0355	·0416	·0477	·0538	·0599	·0661	·0724	43	6	12	18	25	31	37	43	49	55
47	·0724	·0786	·0850	·0913	·0977	·1041	·1106	42	6	13	19	26	32	38	45	51	57
48	·1106	·1171	·1237	·1303	·1369	·1436	·1504	41	7	13	20	27	33	40	46	53	60
49	·1504	·1571	·1640	·1708	·1778	·1847	·1918	40°	7	14	21	28	34	41	48	55	62
50°	1·1918	1·1988	1·2059	1·2131	1·2203	1·2276	1·2349	39	7	14	22	29	36	43	50	58	65
51	·2349	·2423	·2497	·2572	·2647	·2723	·2799	38	8	15	23	30	38	45	53	60	68
52	·2799	·2876	·2954	·3032	·3111	·3190	·3270	37	8	16	24	31	39	47	55	63	71
53	·3270	·3351	·3432	·3514	·3597	·3680	·3764	36	8	16	25	33	41	49	58	66	74
54	·3764	·3848	·3934	·4019	·4106	·4193	·4281	35	9	17	26	35	43	52	60	69	78
55	1·4281	1·4370	1·4460	1·4550	1·4641	1·4733	1·4826	34	9	18	27	36	45	54	63	73	82
56	·4826	·4919	·5013	·5108	·5204	·5301	·5399	33	10	19	29	38	48	57	67	76	86
57	·5399	·5497	·5597	·5697	·5798	·5900	·6003	32	10	20	30	40	50	60	71	81	91
58	·6003	·6107	·6212	·6319	·6426	·6534	·6643	31	11	21	32	43	53	64	75	85	96
59	·6643	·6753	·6864	·6977	·7090	·7205	·7321	30°	11	23	34	45	56	68	79	90	102
60°	1·732	1·744	1·756	1·767	1·780	1·792	1·804	29	1	2	4	5	6	7	8	10	11
61	1·804	1·816	1·829	1·842	1·855	1·868	1·881	28	1	3	4	5	6	8	9	10	12
62	1·881	1·894	1·907	1·921	1·935	1·949	1·963	27	1	3	4	5	7	8	10	11	12
63	1·963	1·977	1·991	2·006	2·020	2·035	2·050	26	1	3	4	6	7	9	10	12	13
64	2·050	2·066	2·081	2·097	2·112	2·128	2·145	25	2	3	5	6	8	9	11	13	14
65	2·145	2·161	2·177	2·194	2·211	2·229	2·246	24	2	3	5	7	8	10	12	14	15
66	2·246	2·264	2·282	2·300	2·318	2·337	2·356	23	2	4	5	7	9	11	13	15	16
67	2·356	2·375	2·394	2·414	2·434	2·455	2·475	22	2	4	6	8	10	12	14	16	18
68	2·475	2·496	2·517	2·539	2·560	2·583	2·605	21	2	4	6	9	11	13	15	17	20
69	2·605	2·628	2·651	2·675	2·699	2·723	2·747	20°	2	5	7	9	12	14	17	19	21
70°	2·747	2·773	2·798	2·824	2·850	2·877	2·904	19	3	5	8	10	13	16	18	21	23
71	2·904	2·932	2·960	2·989	3·018	3·047	3·078	18	3	6	9	12	14	17	20	23	26
72	3·078	3·108	3·140	3·172	3·204	3·237	3·271	17	3	6	10	13	16	19	23	26	29
73	3·271	3·305	3·340	3·376	3·412	3·450	3·487	16	4	7	11	14	18	22	25	29	32
74	3·487	3·526	3·566	3·606	3·647	3·689	3·732	15	4	8	12	16	20	24	29	33	37
75	3·732	3·776	3·821	3·867	3·914	3·962	4·011	14	5	9	14	19	23	28	33	37	42
76	4·011	4·061	4·113	4·165	4·219	4·275	4·331	13	5	11	16	21	27	32	37	43	48
77	4·331	4·390	4·449	4·511	4·574	4·638	4·705	12	6	12	19	25	31	37	44	50	56
78	4·705	4·773	4·843	4·915	4·989	5·066	5·145	11	7	15	22	29	37	44	51	59	66
79	5·145	5·226	5·309	5·396	5·485	5·576	5·671	10°	9	18	26	35	44	53	61	70	79
80°	5·671	5·769	5·871	5·976	6·084	6·197	6·314	9									
81	6·314	6·435	6·561	6·691	6·827	6·968	7·115	8									
82	7·115	7·269	7·429	7·596	7·770	7·953	8·144	7									
83	8·144	8·345	8·556	8·777	9·010	9·255	9·514	6									
84	9·514	9·788	10·078	10·385	10·712	11·059	11·430	5									
85	11·43	11·83	12·25	12·71	13·20	13·73	14·30	4									
86	14·30	14·92	15·60	16·35	17·17	18·07	19·08	3									
87	19·08	20·21	21·47	22·90	24·54	26·43	28·64	2									
88	28·64	31·24	34·37	38·19	42·96	49·10	57·29	1									
89	57·29	68·75	85·94	114·59	171·89	343·77	∞	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

The differences change so rapidly here that they cannot be tabulated.

The cotangent of a small angle of *n* minutes of arc or the tangent of 90° minus *n* minutes is very nearly equal to 3438 divided by *n*.

NUMERICAL ANSWERS

EXERCISE 1.

1.	2.	3.	4.	5.	6.
$OC = 4{\cdot}7$ in.	5·3 in.	5 0 in.	$5\frac{1}{4}$ in.	$5\frac{15}{16}$ in.	5 in.
$AC = 2{\cdot}4$ in.	2·2 in.	3·2 in.	$3\frac{3}{8}$ in.	$2\frac{13}{16}$ in.	$2\frac{1}{2}$ in.

11. 6·1 cm. 12. 4·8 cm. 13. 5·3 cm. 14. 8·4 cm.
15. 7·3 cm. 16. 3·6 cm. 17. 3·7 in. 18. 3·5 in.
19. 4·8 in. 20. 6·1 in. 21. 4·3 in. 22 5·5 in.
23. 8 ft. 5 in. 24. 15 cm. 2 mm. 25. 3 cm. 35. 6 squares.
36. 2 squares, 4 rectangles. 37. 1 square, 4 triangles.
38. 1 circle, 1 curved surface. 39. 2 circles, 1 curved surface.
40. 1 curved surface.

EXERCISE 2.

1. 143°, 180°. 2. 65°, 180°. 3. 2 rt ∠s or 180°.
4. 180°. 6. 58°, 360°. 7. 4 rt ∠s or 360°.
8. 180°, 2 rt ∠s. 9. 15°, $\frac{1}{6}$ rt ∠. 10. 45°, $\frac{1}{2}$ rt ∠.
11. $112\frac{1}{2}$°. 12. 45°. 13. 133°, 47°, 133°.
14. 52°, 128°, 52°. 15. 90°, 90°, 90°.

EXERCISE 3.

5. 90°. 7. 90°. 12. 2. 14. 4.

EXERCISE 5.

1. 180°. 2. 180°. 3. 180°.

EXERCISE 6.

6. 60°. 7. 34°. 8. Impossible. 9. Impossible.
10. 30°, 60°, 90°. 11. 78°. 12. 36°, 72°, 72°. 13. 45°, 45°, 90°.
15. 110°. 16. 110°. 17. 152°. 18. 90°.

EXERCISE 7.

1. 360°. 2. 132°. 3. 110°. 4. 36°, 72°, 108°, 144°.
5. 60°. 7. 4 rt ∠s. 8. 4 rt ∠s. 9. 16 rt ∠s.
10. 44 rt ∠s. 11. 108°. 12. 120°. 13. $128\frac{4}{7}$°.
14. 144°. 15. $176\frac{2}{5}$°. 16. 177°. 17. 4 sides.
18. 5 sides. 19. 12 sides. 20. 150 sides.

EXERCISE 8 (*a*).

1. 6 edges. 2. 36°. 3. 60°. 5. 2 triangles.
7. 4·1 in. 8. 108°.

EXERCISE 8 (*b*).

2. 6. 4. 81°.

EXERCISE 9 (*a*).

1. 8 vertices. 2. 60°. 4. 49°. 7. 61° 10′.

EXERCISE 9 (*b*).

1. 90°. 5. 105°. 6. 5 vertices.
8. 108° is not a factor of 360°

EXERCISE 10 (*a*).

2. 60°. 3. 150°. 5. $(x+60)°$.
6. 55·5 mm. 7. 15 rt ∠s. 8. 61°.

EXERCISE 10 (*b*).

2. 2·3 in. 3. 50°. 4. 10 sides.

EXERCISE 11 (*a*).

2. 69°. 4. 60°. 6. 72°. 7. 120°.

EXERCISE 11 (*b*).

1. 1·45 in. 2. 80°. 7. 137°.

EXERCISE 12.

1. 2·5 in.; 76°, 55°. 2. 4·7 in.; 35°, 43°. 3. 6·9 cm.; 72°, 63°.

EXERCISE 16.

22. 4·95 cm. approximately. 26. 490 ft. 27. 424 ft., 848 ft.

EXERCISE 17 (*a*).

3. 45°. 5. $x=30°$. 7. 144°.

EXERCISE 17 (*b*).

6. 30°. 8. 110°.

EXERCISE 18 (*a*).

2. 2·95 cm. 3. 30°. 4. 12 pairs. 5. $22\frac{1}{2}$°.
6. 180°. 7. 24.

EXERCISE 18 (*b*).

7. 28°.

EXERCISE 19 (*a*).

2. 1 : 2. 4. 120°. 5. 93°. 7. $1\frac{1}{4}$ rt $\angle$s. 8. 23·4 cm.

EXERCISE 19 (*b*).

5. 72°.

EXERCISE 20 (*a*).

3. 75°. 4. 18. 6. 3·57 miles N.; 1·2 miles E. 7. 20.

EXERCISE 20 (*b*).

6. 60°.

EXERCISE 21.

20. $\angle ABC = 75°$, $\angle ABD = 45°$. 21. 4·1 in. 25. 1·6 miles.
27. ·2 in. 28. $AF = 3·25$ in., $\angle BAF = 10\frac{1}{2}°$, $\angle CDE = 105°$.

EXERCISE 22.

1. 8·6 sq. in.	2. 66·4 sq. cm.	3. 11 sq. ft. 59 sq. in.
4. 8593 sq. cm.	5. 16·24 sq. in.	6. 47·25 sq. cm.
7. 6·75 sq. in.	8. 14·04 sq. in.	9. 3·75 sq. in.
10. 20 sq. cm.	11. 5·6 sq. in.	12. 28·5 sq. cm.
13. 5·3 sq. in.	14. 4·3 sq. in.	15. 34·1 sq. cm.
16. 2·6 sq. in.	17. 5·5 sq. in.	26. 40 acres.
27. 44 ac. 3740 sq. yd.	28. ·91 acre.	29. ·735 acre.
30. ·5336 acre.	31. 32·8 lb.	32. 3 sq. m.
33. 975 sq. ft.		

EXERCISE 23 (*a*).

1. 3·9 in. 2. $\frac{1}{2}$ in. 3. $\sin B = \frac{3}{5}$, $\cos B = \frac{4}{5}$, $\tan B = \frac{3}{4}$.
4. $\cos A = \frac{3}{5}$, $\sin A = \frac{4}{5}$, $\tan A = \frac{4}{3}$. 6. $AB = 2$ in.
7. $\sin 71° = ·9455$; $\sin 52° = ·7880$; $\sin 18° = 3090$.
8. $\cos 21° = ·9336$; $\cos 43° = ·7314$; $\cos 85° = ·0872$.
9. $\tan 45° = 1$; $\tan 63° = 1·963$; $\tan 78° = 4·705$. 10. $A = 39°$.

EXERCISE 23 (*b*).

2. From tables, $\sin 56° = \cdot 8290$. 3. $\cos 73° = \cdot 2924$.
4. $\tan 15° = \cdot 2679$. 5. 22·6 ft.
6. 1·333. 7. $\sin 60° = \cdot 8660$; $\cos 60° = \cdot 5$; $\tan 60° = 1\cdot 7320$.
8. $\frac{12}{13}$ or ·923. 9. 119·8 ft. 10. 139·88 ft.
11. 51° 21′. 12. 61° 56′; 119° 4′. 13. $\angle B = 36° 52'$; $\cos B = \cdot 8$; $AB = 5$.

EXERCISE 24.

1. $c = 6\cdot 5$ in. 2. $b = 2$ in. 3. $a = 3$ in.
4. $a = 3\cdot 5$ in. 5. $c = 13\cdot 7$ cm. 6. $c = 12$ in.
7. $a = 2\cdot 1$ cm. 8. $c = 4\cdot 2$ in. 9. 385·8 mm.
10. $a = 9\cdot 3$ in. 11. $\sqrt{3} = 1\cdot 73$. 12. $\sqrt{5} = 2\cdot 24$.
13. $\sqrt{7} = 2\cdot 65$. 14. $\sqrt{11} = 3\cdot 32$. 15. 32 ft.
16. 18·4 cm. 17. 7·1 in. 18. 21 ft.
19. 6 in. 20. 17·6 in. 21. (*a*), (*c*).
23. 4·7 in. 26. $AC^2 = 1296 + 1156 = 2452$; $EC^2 = 2916 + 2601 = 5517$; $AE^2 = 225 + 7744 = 7969$; $AC^2 + EC^2 = 2452 + 5517 = 7969$.
27. $\sqrt{34}$ or 5·83. 28. $\sqrt{98}$ or 9·9. 29. Yes; 12·04 in.
30. $2\frac{1}{2}$ in. 31. 48 ft. 32. 45°, 45°, 90°; ·5 sq. in.
33. 337·5 yd. 34. 68 ft. 35. 155·5 yd.

EXERCISE 25 (*a*).

2. 2 in. 3. 11 sq. in. 7. 65°.

EXERCISE 25 (*b*).

1. 5·24 sq. in. 2. 35 sq. cm. 7. 4 in.

EXERCISE 26 (*a*).

1. 3 ways. $A = \frac{1}{2}bh$. 2. Ratio $= 3 : 4 : 5$; $5^2 = 4^2 + 3^2$.
3. 8 sq. in. 6. 113°. 8. 15°.

EXERCISE 26 (*b*).

1. 2·85 sq. in. 2. 10 sq. in. 6. 1·41 in.

EXERCISE 27 (*a*).

1. $\frac{1}{2}d(x+y)$. 2. 12 sq. in. 3. 6·9 sq. in. 7. 125°.

EXERCISE 27 (*b*).

1. 36·2 sq. cm. 2. 13·7 sq. in. 5. 5·66 in.
6. 1·03 in. 8. 177°.

EXERCISE 28 (*a*).

1. $A=\sqrt{s(s-a)(s-b)(s-c)}$. 2. 2 sq. in. 3. 5 cm.
7. 70°.

EXERCISE 28 (*b*).

1. 6·4 sq. in. 2. 24 sq. in. 5. 4·95 cm.; 11·88 sq. cm.
8. $x=10°$.

EXERCISE 29.

1. $6\frac{1}{2}$ in. 2. 15·26 cm. 3. 32·25 cm. 4. 15 sq. in.
6. 1·66 in. 11. 9·43 in. 12. 3·32 in.

EXERCISE 30.

1. $\angle OAB=39°$; Reflex $\angle AOB=258°$. 2. 51°. 3. 129°.
4. Reflex $\angle AOB=216°$; $\angle ARB=108°$; $\angle OAB=18°$.
5. $\angle OAB=31°$; $\angle OBC=59°$. 6. 63°. 7. 122°.
8. 105°. 9. 109°, 91° 15. 3·46 sq. in. or $2\sqrt{3}$.

EXERCISE 31.

2. 2·71 in. 3. 1·6 in. 4. 2·62 in. 5. 2·33 in.
6. 6 in. 9. 2·64 in. 12. 2·29 in. 14. 2·23 in.
15. 6·71 in. 16. 7·1 in. 17. 9 ft. 11·8 in. 18. 5 ft. 1·6 in.
19. 6 cm. 21. 3·46 in. 22. $DE=3·25$ in.

EXERCISE 32.

1. 88 in. or 7 ft. 4 in. 2. 26·4 in. 3. 44 ft.
4. $100\frac{4}{7}$ ft. 5. $97\frac{3}{7}$ ft. 6. 41·36 in.
7. 94·25 in. 8. 11·62 in. 9. 229·34 ft.
10. $r=2·82$ ft.; $d=5·64$ ft. 11. $r=2·46$ ft.; $d=4·91$ ft.
12. $r=6·84$ in.; $d=13·68$ in. 13. $r=1·23$ in.; $d=2·45$ in.
14. $r=·23$ ft.; $d=·45$ ft. 15. $r=·17$ in.; $d=·33$ in.
16. 66·4 sq. in. 17. $r=1·24$ in.
18. 58·5 lb. 19. $56\frac{4}{7}$ sq. cm.
20. 6 ft. 2 in. 21. $14\frac{1}{7}$ in.

EXERCISE 33 (*a*).

1. 63°. 2. 6 in. 5. 20°. 6. Two.
7. 48°, 60°, 72°. 8. 60°.

EXERCISE 33 (*b*).

1. 120°. 2. $AC = 2{\cdot}24$ in., $\angle ABC = 45^\circ$. 3. 40 ft. 4. 6·3504 sq. in.

EXERCISE 34 (*a*).

1. 10 cm. 2. $t = \sqrt{c^2 - r^2}$. 4. 30°. 5. 152°, 72°, 136°.
7. 150°. 8. 30 sq. in.

EXERCISE 34 (*b*).

3. About $57\frac{1}{2}^\circ$.
6. Distance $= 212{\cdot}5$ ft., $\angle$ of elevation = between $24\frac{1}{2}^\circ$ and 25°.
7. ·81 in.

EXERCISE 35 (*a*).

1. 8 cm. 2. 12 sq. cm. 5. 4. 6. 13 in.
7. 8 N., 5 W. 8. 20 sq. in.

EXERCISE 35 (*b*).

1. 36°. 4. 12·7 cm. 6. ·8 mile. 8. 4 in., 4 in., 5·7 in.

EXERCISE 36 (*a*).

1. $\angle AOC = 40^\circ$; $\angle OAC = 70^\circ$. 2. 3·1416. 4. 30°.
6. 105°. 7. 2 in. 8. 4; 2 on each side of the base.

EXERCISE 36 (*b*).

1. 150°. 3. Other side $= 7{\cdot}42$ in., area $= 11{\cdot}1$ sq. in. 6. 2·12 in.

EXERCISE 37 (*a*).

3. $T = \sqrt{c^2 - (a-b)^2}$. 4. Equal to the radius.
6. $\sqrt{3}$ sq. in., or 1·73 sq. in. 7. 6 in. 8. 4 cm.

EXERCISE 37 (*b*).

1. 3·07 in.

EXERCISE 38 (*a*).

1. $\angle BOD = 144^\circ$, $\angle COD = 72^\circ$. 5. 10 miles. 6. 36°.
7. 38°. 8. 10·38 sq. in.

EXERCISE 38 (*b*).

1. $AB = 1{\cdot}76$ in., $BC = 1{\cdot}09$ in., $CA = -2{\cdot}85$ in.;
   $AB + BC + CA = 0$; $CA = -2{\cdot}61x$.
3. $29{\cdot}2$ ft. 4. $\angle BAC = 65°$, $\angle AEB = 84°$, $\angle CEB = 96°$.

EXERCISE 39 (*a*).

1. 124°. 2. 150°. 6. $x = 45°$. 8. 12 sq. units.

EXERCISE 39 (*b*).

1. $\angle OQP = 1$ rt. $\angle$, $TQ = QP$.
2. $\angle A =$ about $63{\cdot}8°$; $Bx = 3{\cdot}4$ cm.; $Bx = \frac{1}{2} AC$.

PRINTED
BY
WALTER LEWIS M A
AT
THE CAMBRIDGE
UNIVERSITY
PRESS